E Trigonometric Substitutions.

E.1 The Area of a Circle.

The area of a circle of radius 1 (the unit circle) is well-known to be π. We will investigate this with several different approaches, each illuminating a particular aspect of the calculus. Remember that the unit circle is described by the equation

$$x^2 + y^2 = 1.$$

We'll begin by finding the area of the region contained in the quarter circle between the circle, the X-axis, and the Y-axis. This area can be found using a Riemann integral, namely:

$$\int_0^1 \sqrt{1-x^2}\, dx.$$

We'll first show that this integral is $\pi/4$ using a few elementary ideas and a convergent improper integral. After this rather tricky approach we'll examine the same problem from a more elementary point of view. From this we'll develop another integral that finds the area of the circle , leading us to a sometimes useful method of integration based on a substitution of trigonometric functions .

Theorem 1.

$$\int_0^1 \sqrt{1-x^2}\, dx \;\; = \pi/4.$$

Proof: The key idea is to make a substitution, $u = \sqrt{1-x^2}$ so $u^2 = 1-x^2$ and $2u\; du = -2x\, dx$. When $x = 0$, $u = 1$, while when $x = 1$, $u = 0$. After the substitution we have an improper integral. (We'll ignore this difficulty for now and continue assuming that the integral converges.) Thus

$$\int_0^1 \sqrt{1-x^2}\, dx \;\; = \;\; -\int_1^0 \frac{u^2}{\sqrt{1-u^2}}\, du$$

Now we can change the order of integration of this last integral and rename the variable u to x to obtain

$$\int_0^1 \sqrt{1-x^2}\,dx \;=\; \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx.$$

Combining these integrals and placing the integrand over the common denominator of $\sqrt{1-x^2}$ gives

$$\int_0^1 \frac{1-2x^2}{\sqrt{1-x^2}}\,dx \;=\; 0.$$

But now we have

$$2\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx \;=\; \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx \;= \arcsin(1) \;= \pi/2.$$

Thus we have finally that

$$\int_0^1 \sqrt{1-x^2}\,dx \;=\; \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx \;=\; \pi/4.$$

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Comments: You should check that in these calculations the improper integrals that appeared were all used properly! You should now be able to show that the area of a circle with radius r is $\pi\, r^2$. In this way we see something that was known as far back in history as the Greek mathematician Archimedes. Namely that the area of any circle is equal to the area of a right triangle in which one side adjacent to the right angle is the radius of the circle and the other side is the circumference of the circle.

E.2 Another Definite Integral That Finds the Area of the Circle.

We have seen that the area of one quarter of the unit circle is expressed as

$$\int_0^1 \sqrt{1-x^2}\,dx \;=\; \pi/4.$$

To estimate this integral we could partition the interval [0,1] with n+1 points, $x_k = k/n$, where $k = 0$ to n. The Euler sum estimate would be

$$\sum_{k=0}^{n-1} \sqrt{1 - x_k^2}\ \Delta x_k$$

where $\Delta x_k = 1/n$. Examining the points on the unit circle that corresponds to this partition, namely (x_k, y_k) where $y_k = \sqrt{1 - x_k^2}$ we let t_k be the angle measured in radians from the point (1,0) on the unit circle so that $x_k = cos(t_k)$ and $y_k = \sqrt{1 - x_k^2} = sin(t_k)$. Using differentials we have that when $x = cos(t)$, $dx = sin(t)\ dt$. Notice that when $x = 0$, $t = \pi/2$ while when $x = 1$, $t = 0$.

If we replace all the x's in the Euler sum with the appropriate term using the angles t_k we arrive at the following sum:

$$\sum_{k=0}^{n-1} sin(t_k) \cdot -sin(t_k)\ \Delta t_k\ .$$

But when you analyze this sum carefully, it should become clear that it is a Reimann sum for the following integral:

$$\int_{\pi/2}^{0} -sin^2(t)\ dt\ .$$

After a change in sign that goes with changing the order of the limits of integration for this integral we have that the area of one fourth the unit circle can be found by evaluation of

$$\int_{0}^{\pi/2} sin^2(t)\ dt\ .$$

Using the methods discussed in the section on trigonometric integrals or integration by parts, you can show that this integral is $\pi/4$.

Comments. We could use the method we established here to again find the area of a circle of radius r, but we leave that for you as an exercise.

It is not so surprising to find π in this answer since it involved in the limits of the integral, arising as the value of t for which $cos(t) = 0$.

E.3

Trigonometric Substitutions Using the Sine Function.

Although the last work on the area of the circle suggests a method of substitution in an integral involving $\sqrt{1-x^2}$ in the integrand, it was a little awkward to follow through on this because of the reversal of signs in both the differential and the order of integration of the definite integral. For this reason the method used to simplify these kinds of integrals uses the substitution that $x = sin(\theta)$ with the result that $dx = cos(\theta)\ d\theta$. This substitution is fairly straight forward in both indefinite and definite integrals.

In particular, the expression $\sqrt{1-x^2} = \sqrt{1-sin^2(\theta)} = \sqrt{cos^2(\theta)} = cos(\theta)$.

Example VII.E.1 Find $\int \sqrt{1-x^2}\ dx$ using the trigonometric substitution $x = sin(\theta)$.

Solution. Let $x = sin(\theta)$ so $dx = cos(\theta)\ d\theta$. Continuing we have

$$\int \sqrt{1-x^2}\ dx = \int cos(\theta)\ cos(\theta)\ d\theta = \int cos^2(\theta)\ d\theta.$$

From our previous work on trigonometric integrals this last integral can be shown to be $1/2\ [sin(\theta)\ cos(\theta) + \theta] + C$. But our problem was originally posed in terms of x so we must express our answer in terms of x. This is easy for $sin(\theta)^{-} = x$ and with a little more reflection on how this substitution worked it must be that $cos(\theta) = \sqrt{1-x^2}$ while θ itself must be the arcsin(x). Putting this all together we see that

$$\int \sqrt{1-x^2}\ dx = 1/2\ [x\ \sqrt{1-x^2} + \arcsin(x)\ ] + C.$$

Comment. Using the last example we apply the Fundamental Theorem of Calculus to evaluate our now familiar integral for the area of the quarter unit circle to find once again that

$$\int \sqrt{1-x^2}\ dx \ = \frac{1}{2}\ [x\ \sqrt{1-x^2}\ +\ \arcsin(x)\ ]|_0^1$$

$$= \frac{1}{2}\ \arcsin(1)\ \ = \pi/4.$$

Here are two more examples of how the sine function can be used in a substitution to convert an algebraic integral to a trigonometric integration problem.

Example VII.E.2. Find $\int\ \sqrt{9-x^2}\ dx$.

Solution. Our first question is what to use for x in this integral that will eliminate the expression $\sqrt{9-x^2}$ after the substitution is accomplished. From the last example we see that the key trionometric identity for this problem is $1 - sin^2(\theta) = cos^2(\theta)$ but in our problem we have to work with the expression $9 - x^2$. So we want $9 - x^2 = k(\ 1 - sin^2(\theta)\ )$ for some value of k. It is not too hard to see from this that $k = 9$ so that $x^2 = 9\ sin^2(\theta)$ and therefore the sensible choice for this problem is to use the trigonometric substitution $x \ = \ 3\ sin(\theta)$. Then $dx \ = \ 3\ cos(\theta)\ d\theta$.

Continuing we have

$$\int\ \sqrt{9-x^2}\ dx\ \ = \int\ 3\ cos(\theta)\ 3\ cos(\theta)\ d\theta\ \ = \int\ 9\ cos^2(\theta)\ d\theta.$$

From our previous work on trigonometric integrals or integration by parts, this last integral can be shown to be $9\ \frac{1}{2}\ [sin(\theta)\ cos(\theta) + \theta]\ +\ C$. But our problem was originally posed in terms of x so as usual we must express our answer in terms of x. This is easy for $sin(\theta)\ =\ x/3$ and with a little more reflection on how this substitution worked it must be that $cos(\theta)\ =\ \frac{\sqrt{9-x^2}}{3}$ while θ itself must be the $\arcsin(x/3)$. Putting this all together we see that

$$\int\ \sqrt{9-x^2}\ dx\ \ =\ \ 9/2\ [\frac{x\ \sqrt{9-x^2}}{9}\ \ + \arcsin(x/3)\ ] +\ C.$$

$$= \frac{x\,\sqrt{9-x^2}}{2} \;+\; \frac{9}{2}\;\arcsin(x/3) \;+\; C.$$

In the next example we'll see how a quadratic expression in x involved with a square root can also lead to a substitution involving the sine function.

Example VII.E.3. Find a trigonometric substitution for $\int\sqrt{5+4\mathrm{x}-\mathrm{x}^2}$ dx that will eliminate the square root in the intregral after the substitution is made.

Remark. From our previous work on trigonometric integrals and/or integration by parts this last integration is not difficult. As usual we would still need to express our answer in terms of x. This is fairly direct since $sin(\theta) \;=\; \frac{x-2}{3}$and with a little more reflection on how this substitution worked it must be that $cos(\theta) \;=\; \frac{\sqrt{5+4x-x^2}}{3}$ while θ itself must be the $\arcsin(\frac{x-2}{3})$. You may complete the details of the work now to find the integral in terms of x in the last example.

E.4

Trigonometric Substitutions Using the Tangent and Secant Functions.

Example VII.E.4 Find $\int\;\frac{1}{x\sqrt{x^2-1}}\;dx$ using the trigonometric substitution $x \;=\; sec(\theta)$.

Remark. You might check this result by reviewing the methods we considered for finding the derivative of inverse (trigonometric) functions.

Example VII.E.5 Find $\int\;\frac{1}{\sqrt{1+x^2}}\;dx$ using the trigonometric substitution $x \;=\; tan(\theta)$.

Solution. Let $x \;=\; tan(\theta)$ so $dx \;= \sec^2(\theta)\;d\theta$. Continuing we have

$$\int\frac{1}{\sqrt{1+x^2}}\,dx \;=\int\;\frac{sec^2(\theta)}{sec(\theta)}\,d\theta = \int\sec(\theta)\;d\theta = \ln(|\sec(\theta)+\tan(\theta)|)+C.$$

$$\int\;\frac{1}{\sqrt{1+x^2}}\;dx = \ln(|\sqrt{1+x^2}+x|)+C.$$

Comments.1.This last example illustrates a rather interesting feature of this method of substitution. Neither the integrand nor the solution to the problem involve any explicit use of trigonometric functions, while the method we used to find this result relied heavily on trigonometry. There is a result in logic (and philosophy) that says that if a problem and it's solution do not involve a particular concept, then there is a way to arrive at that solution using only concepts contained already in the problem and it's solution. This result is called "Craig's Lemma". In this example the implication of this result is that there should be a demonstration of the truth of this solution that does not requiretrigonometry. Of course this is true, since to check the integral is correct here you need only differentiate and in that work you will not use any trigonometry.

We need to express our answer in terms of

2. If we consider the learning model with $L'(t) = 1/\sqrt{t^2+1}$ and $L(1) = 0$ as usual, the last integration shows that this is another example of unbounded learning . See the exercises for applications of these integrals to other models.

E.5 Exercises.

Evaluate the following integrals as appropriate:

31-40. For each of the integrals in exercises 1 to 10 sketch a tangent field for the appropriate differential equation. Discuss briefly the domain of the solution and sketch two integral curves on your sketch.

41. Suppose a population P grows so that $P'(t) = \sqrt{1+t^2}$ with $P(0) = 1$. Estimate $P(1)$ using Euler's method with $n = 4$, $n = 10$, and $n = 100$. Find a solution to this differential equation and estimate the population when $t = 10$ and $t = 100$.