Chapter V.G The Definite Integral and Area: Two Views

In this section we will look at the application of the definite integral to the problem of finding the area of a region in the plane. In particular the regions in our discussion are bounded by either lines or the graphs of functions. There are two ways to approach this problem as in many other applications of the definite integral. We will consider the area problem in some detail because it is typical.

Our first approach for finding an area will center on the approximation of the area as a sum of areas of rectangles that are determined by the given region. This follows our earlier treatment of the definite integral as a limit of sums: What's good for one is good for all and the more the merrier. These estimating sums will have a limiting value which by definition is a definite integral. A second contrasting approach to the same problem follows the theme that information about a rate of change (a differential equation) can help determine the value of a variable which might otherwise be considered constant. In this approach we'll consider the area as a function of a single variable, much like in the derivative form of the fundamental theorem of calculus. The key here is to estimate the rate at which the area is changing as a function of one variable. The area of a rectangle forms the basis for an estimate of the change in area of the region.

1. Area as a Limit of Sums

Our initial work on the definition of the definite integral used an interpretation of the Euler sums as the area of rectangles which approximated the area of a region enclosed by the graph of a non-negative function, the X -axis , and lines X = a and X = b. The next example generalizes this situation. Keep in mind the definition of the definite integral, namely:

$$\lim_{\Delta x \to 0} \sum_{k=1}^{k=N} P(x_{k-1})\Delta x = \int_a^b P(x)\,dx$$

This definition allows us to recognize the limit of estimates of this form as definite integrals. The advantage of seeing this form is that the definite integral in many cases can be evaluated exactly by using the Fundamental Theorem of Calculus. This connection will be more clear after you read the next example.

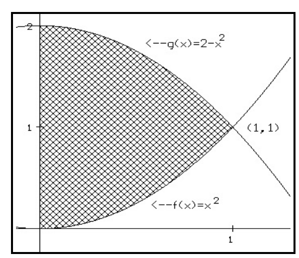

Figure 1

Example V.G.1. In this example we will a) express the area of this region as the difference of two areas; b) estimate the area of the region in the first quadrant enclosed by the Y-axis, the graph of $g(x)=x^2$ and $f(x)=2-x^2$; and c) express the area of this region as a definite integral, explaining how this integral is related to the difference in part a) and the estimate in part b).

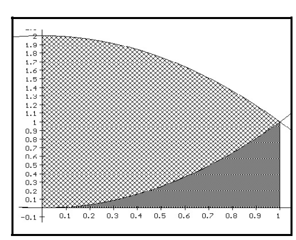
Figure 2

Discussion: a) First graph the region as described and note that the curves intersect in the first quadrant when $x^2=2-x^2$, i.e., when x = 1, so the x values for this region are between 0 and 1. See Figure ***.The region can be considered as being obtained from the region beneath the graph of $Y = 2 - X^2$ and above the X-axis between the Y - axis and the line X = 1, with area A1, after cutting away the region beneath the graph of

$Y = X^2$ and above the X-axis between the Y-axis and the line X = 1, with area A2.

The area can therefore be described as the difference of the areas of the two regions, i.e., A1-A2. Each of these areas can be computed from a definite integral, $A1 = \int_0^1 2 - x^2 dx = 2x - \frac{x^3}{3}\Big|_0^1 = 2 - \frac{1}{3} = \frac{5}{3}$ and $A2 = \int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$. So the area we want to find is 5/3 - 1/3 = 4/3.

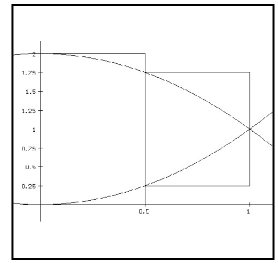

Figure 3

b) For a first estimate we cut the interval in half and estimate the area by the areas of two rectangles with base 1/2 and altitudes determined by left hand endpoints, 0 and 1/2. With these heights of 2=2-0 and 7/4 - 1/4 = 3/2, the estimate for the area is 2· 1/2+ 3/2· 1/2 = 7/4. See Figure ***.

Of course a finer partition of the interval and a correspondingly larger number of rectangles would give a better estimate of the area. In general if we let $\Delta x = 1/N$ and $x_k = k/N$, then the estimate for the area is given by $\sum_{k=1}^{N} (2 - x_{k-1}^2) - (x_{k-1}^2)\,\Delta x = \sum_{k=1}^{N} (2 - 2x_{k-1}^2)\,\Delta x$. Larger values for N will give a better estimate for the area as Figure *** illustrates. With N = 10 we find an estimate for the area of 1.43. Notice that from the figure it should clear that any of the estimates obtained using these sums will be an overestimate of the actual area.

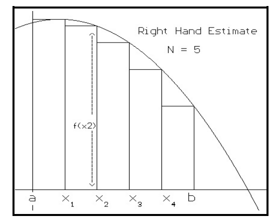

Figure 4

c)The areas in part a) were expressed as integrals, $A1 = \int_0^1 2 - x^2 dx$ and $A2 = \int_0^1 x^2\, dx$ while the area in question was the difference A1-A2.. This area can be expressed also as a single integral by using the linearity property of the definite integral, $A1 - A2 = \int_0^1 2 - x^2\, dx - \int_0^1 x^2\, dx = \int_0^1 2 - 2x^2\, dx$. Notice that this is precisely the definite integral that the sum in part b) estimates by the definition of the integral. Thus we have two approaches to finding the area: i) recognize the region as being composed of regions for which the areas are computable and ii) estimate the area with sums. With each approach the area can be found by evaluating a single definite integral using the Fundamental Theorem of Calculus, $\int_0^1 2 - 2x^2\, dx = 2x - \frac{2}{3}x^3\Big|_0^1 = \frac{4}{3}$.

Generalization. The work in the last example can be generalized without much difficulty if we follow the estimation approach but continue to **assume that for all x in [a,b], g(x)≤f(x)**. In this situation the region will be contained between the graphs of y=f(x) and y=g(x) and the lines X = a and X = b with a < b and where f and g are continuous functions on [a,b]. We partition the interval [a,b] into N pieces of length $\Delta x = \frac{b-a}{N}$ with $x_k = a + k\,\Delta x$. Above the k^{th} subinterval we consider a rectangle with its height h_k determined by the difference in the function values at the left hand endpoint of the interval, i.e. , $h_k = f(x_{k-1}) - g(x_{k-1})$. The sum of areas of the individual rectangles, given by $\sum_{k=1}^{N} h_k \Delta x = \sum_{k=1}^{N} f(x_{k-1}) - g(x_{k-1})\Delta x$, is our estimate for the area of the region. As $N \to \infty$, these sums give better estimates for the area of the region, but they also approach (by definition) a definite integral. Therefore the area of the region must be given by

$\int_a^b f(x) - g(x)\, dx$.

Comment. In many situations the functions whose graphs bound a region change over the controlling domain interval. It is not uncommon in fact for two function's graphs to cross so that for a part of the interval f(x) $\le$ g(x) while for another part g(x)$\le$ f(x). In these situations the region needs to be broken down for analysis into regions over intervals where the upper and lower curves do not change. Finding the values for x where these changes happen, for example, where f(x) = g(x), is crucial to this investigation. For convenience it is possible to denote the result in situations where only two functions, f and g, are involved without these complications by noting that $h_k = | f(x_{k-1}) - g(x_{k-1}) |$ in either case. However, in practice this notation does not actually simplify the computation since to evaluate an integral with an absolute value in the integrand requires further analysis of the integrand.

Summary: The area of a region between two continuous functions f and g and the lines X=a and X=b is given by $\int_a^b |f(x) - g(x)|\, dx$.

The next example follows the theme of estimation to find an area. It differs from the previous example primarily because the estimating rectangles are controlled by the use of the vertical axis, giving an alternative for regions where the boundary curves are more easily described with the horizontal (X) component of the points on the curves being deter function of their vertical (Y) component. We continue to follow the principles of What's Good For One Is Good For All and The More The Merrier!

Example V.G.2. Consider the region bounded by the curve $Y^2 = X$ and the line with equation Y = X - 2. a) Estimate the area of this region. b) Express the area of this region as a definite integral. c) Use part b to find the area of the given region.

Solution. a) First we graph the region under discussion and note that the curves intersect when $Y^2 = Y + 2$, i.e., the points on the curve where Y =-1 and Y = 2 so the Y values for this region are between -1 and 2. As a first estimate we cut this interval of Y values [-1,2] on the Y-axis into three equal pieces of length 1 and estimate the area by the areas of three horizontal "rectangles" with width 1 and lengths of 0, 2 and 3 - 1 = 2, giving the estimate of (0 + 2 + 2) 1 = 4.

Of course a finer partition of the interval and a correspondingly larger number of horizontal rectangles would give a better estimate of the area. In general we let $\Delta y = 3/N$ and $y_k = -1 + k\Delta y$. We use the horizontal segment connecting the points on the curves with Y coordinates y_{k-1} to determine the side of the k^{th} rectangle. This gives the k^{th} rectangle dimensions $(y_{k-1}+2) - y_{k-1}^2$ by Δy so then the estimate for the area of the k^{th} rectangle is given by $(y_{k-1}+2) - y_{k-1}^2\ \Delta y$. This leads to an estimate for the total area given by the sum $\sum_{k=1}^N (y_{k-1} + 2 - y_{k-1}^2)\,\Delta y$ or $\sum_{k=1}^N (2 + y_{k-1} - y_{k-1}^2)\,\Delta y$. As you might expect larger values for N will give a better estimate for the area. [Draw a figure that illustrates this situation with rectangles.] For example with N=15, we find an estimate of 4.48 .

b) Here the pattern of estimating an area with sums shows its payoff. You should be able to recognize that the sum in part a) estimates the definite integral: $\int_{-1}^{2} 2+y-y^2\, dy$. Since these sums can estimate only one number as their limit when $N\to\infty$, it must be that the area is the number described by $\int_{-1}^{2} 2+y-y^2\, dy$.

c) From part b) the area can be found by evaluating the definite integral using the Evaluation Form of the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_{-1}^{2} 2+y-y^2\, dy &= 2y+\frac{y^2}{2}-\frac{y^3}{3}\Big|_{-1}^{2}\\ &= (4+2-\frac{8}{3}) - (-2+\frac{1}{2}+\frac{1}{3})\\ &= 4\frac{1}{2}.\end{aligned}$$

Comment: The generalization of this example is to regions that have their boundaries determined by curves described with the X-coordinate of a point on the curve determined as a function of its corresponding Y-coordinate. In the example we had $X=f(Y)=2+Y$ and $X=g(Y)=Y^2$ and an interval of Y values [c,d] which controlled the boundary curves values. The result for regions of this type is that the area can be expressed as $\int_{y=c}^{y=d} |g(y)-f(y)|\, dy$.

2. Area as a Solution to a Differential Equation

In our first treatment of the Fundamental Theorem of Calculus we saw that an area function was the solution to a differential equation. We will find the same kind of analysis useful in finding the area between two curves as the next example illustrates by resolving our first example with a quite different approach using the derivative and the fundamental theorem of calculus for differential equations.

Example V.G.3. a) Let A be the function determined by the statement that A(t) denotes the area of the region in the first quadrant enclosed by the Y - axis, the graph of $g(x)=x^2$, $f(x)=2-x^2$, and the line X = t when t is between 0 and 1. Estimate the instantaneous rate of change of A(t) at t where t is in the interval [0,1].

b) Find the instantaneous rate of change of the area A of this region as a function of t.

c) Explain how the differential equation in part b) shows the area in question must be $\int_0^1 2-2x^2\, dx$.

Solution. a) An estimate of the rate of change at t is given by $\frac{A(t+\Delta t)-A(t)}{\Delta t}$. But $A(t+\Delta t)-A(t)$ is approximately the area of a rectangle with altitude g(t)-f(t) and base Δt, so that

$\frac{A(t+\Delta t)-A(t)}{\Delta t} \approx f(t)-g(t)$. Draw a figure that illustrates this for the functions *f* and *g*.

b) From part a) we have $A'(t) = \lim_{\Delta t \to 0} \frac{A(t+\Delta t)-A(t)}{\Delta t} = f(t)-g(t)$.

c) The initial condition for A(t) is A(0)=0. By the Differential Equation form of the Fundamental Theorem of Calculus, the solution to the differential equation $A'(t)=f(t)-g(t)=2-2t^2$ with A(0)=0 is $\int_0^t 2-2x^2\,dx = 2x-\frac{2}{3}x^3\Big|_0^t = 2t-\frac{2}{3}t^3$.Therefore $A(1)= \int_0^1 2-2x^2\,dx = 2-\frac{2}{3} = \frac{4}{3}$.

Exercises.

In each of the following problems a) sketch a graph of the region described, b) use four rectangles to estimate the area of the region, c) express the area of the region using the definite integral, and d) find the exact area of the region described.

1. The region bounded by the graph of the equations $Y = X^2$ and $Y = X^3$.
2. The region bounded by the graph of the equations $Y = X^2$ and $Y = X$.
3. The region bounded by the graph of the equations $Y = X$ and $Y = X^3$.
4. One section of the region bounded by the graph of the equations Y=sin(X) and Y=cos(X).
5. The region bounded by the graph of $Y = X^2$ and the graph of $Y = X^{1/2}$.
6. The region bounded by the graph of the equations $Y = X$ and $Y = 2 - X^2$.

In each of the following problems a) sketch a graph of the region described, b) use four horizontal rectangles to estimate the area of the region, c) express the area of the region using the definite integral with y as the controlling variable, and d) find the exact area of the region described.

7. The region bounded by the graph of the equations $X=Y^2$, $Y = X$, $Y = -1$ and $Y = 2$.
8. The region bounded by the graph of the equations $X=Y^2$ and $Y = X$, $Y = -2$ and $Y = 2$.
9. The region bounded by the graph of the equations $Y = X$ and $X=Y^2-2Y$.
10. The region bounded by the graph of the equations X=sin(Y), X=cos(Y), Y=0 and Y=π.
11. The region bounded by the graph of $Y=X^2$ and the graph of $Y=X^{1/2}$.
12. The region bounded by the graph of the equations $Y= 1/x$, $Y= 1/x^2$, and the line X=2.