Chapter III.D Other Graphing Applications of The Derivative.

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Graphs are powerful tools for understanding relations between variables. As the last two sections have shown, analysis of the first and second derivatives of a function can give insight into the graph of a function. In those sections we considered primarily the information we could obtain directly about the graphs. We answered such questions as: when is the graph increasing? what is the concavity? where are any extreme values and points of inflection? This type of information contributes to our understanding of other applications by the robust nature of a graph to represent relations. In this section we look at some other graphical features of functions and their relation to the derivative.

Vertical Asymptotes. In our discussion of continuity in Chapter I.I we met situations where a function, f, fails to have a limit because it "blows up," i.e., the function value, $f(x)$, increases without bound as x approaches a specific number (from one or both sides). Physical situations in which this type of functional behavior can be observed directly are not possible since we do not have ability to measure unlimited large quantities. This is a danger of graphing technology: it uses procedures to create the illusion of a curve based on a large but still finite sample of data.

Caution is advised when approaching these enormous but sometimes hidden holes in a graph. Failing to notice blowing up can lead to serious mistakes in the analysis of a function. The first and/or the second derivatives can help with this analysis and graphing of functions near these key points of discontinuity.

Table 1 $x \to 1$

x	$\frac{1}{(x+1)(x-1)^2}$
0.9	52.6315789474
0.99	5025.12562814
0.999	500250.1
1.1	47.619047619
1.01	4975.12437811
1.001	499750.1

Table 2 $x \to -1$

x	$\frac{1}{(x+1)(x-1)^2}$
-0.9	2.77008310249
-0.99	25.2518875786
-0.999	250.250187625
-1.1	-2.26757369615
-1.01	-24.7518625777
-1.001	-249.750187375

Example III.D.1. Consider $f(x) = \dfrac{1}{(x+1)(x-1)^2}$ which is not defined when $x = -1$ and $x = 1$.

When $x \to 1$ (from either the left or the right) $f(x) \to \infty$. See Table 1. On the other hand when $x \downarrow -1$, $f(x) \to \infty$. The limit is just the opposite as $x \uparrow -1$, namely $f(x) \to -\infty$. See Table 2 and Figure III.D.1.

The graph of f is said to have **vertical asymptotes** at $x = -1$ and $x = 1$ because when x is close to these two points, $f(x) \to \pm\infty$. Visually the curve of the graph appears to be getting closer and closer to the vertical lines X = 1 and X = -1.

Now let's examine the derivative:[Here we think of f as the product of $1/(x+1)$ and $1/(x-1)^2$.]

$$f'(x) = \frac{1}{(x-1)^2}\cdot\frac{-1}{(x+1)^2} + \frac{1}{(x+1)}\cdot\frac{-2}{(x-1)^3} = \ldots = \frac{-(3x+1)}{(x-1)^3(x+1)^2}.$$

The function f has no derivative at $x = \pm 1$ and has a critical point at $x = -1/3$. Here then is an analysis of $f'(x)$:

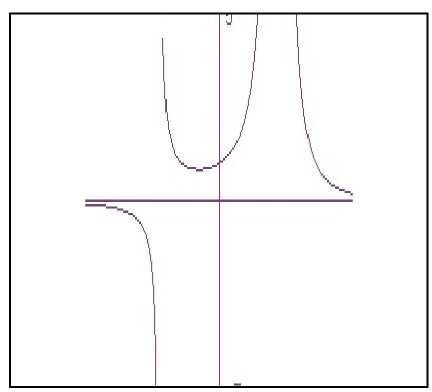

Figure III.D.1

Graph of $f(x) = \dfrac{1}{(x+1)(x-1)^2}$

```
f'(x)    ---T-----0+++++++++++T---------  T=undefined
        _|___|___|___|___|___|___|___|___|_
             -1    -1/3 0        1
  f  is  decr  decr  increasing  decr
            ↘     ↘      ↗         ↘
```

As x approaches the two points where f is not defined the sign of $f'(x)$ indicates precisely how the function's values "blow up".

Thus as x↑-1, the values of f must be decreasing, so f(x)→ -∞, while as x↓-1, the values of f must be decreasing on the interval as we proceed well, so f(x) → ∞. The behavior of f near 1 is also consistent with the first derivative analysis. As x↑1 we have the values of f are increasing so f(x)→∞, while we see that the values of f(x) are decreasing on the interval (-1,-1/3) so as x↓1, f(x)→ +∞.

In summary, this example illustrates that at a vertical asymptote the behavior of the function can be analyzed using the first derivative to see how the function blows up consistently with the increasing and decreasing behavior of the function.

You should pursue these ideas further on your own to write a description of precisely what knowledge about the first derivative's sign does to help with the graphing of vertical asymptotes.

For the sake of simplicity as well as variety, our analysis of these vertical asymptote problems using the second derivative will look at a less complicated function.

Example III.D.2. Let $f(x) = \frac{1}{x-1}$. See Table 3 and Figure III.D.2.

This function has a vertical asymptote at X = 1.

First we find that $f'(x) = \frac{-1}{(x-1)^2}$ and from this that $f''(x) = \frac{2}{(x-1)^3}$.

An analysis of the second derivative here is illustrated by the following figure.

```
f''(x)----------T++++++++   T=undefined
      __|____|____|____|____|____|__
        0         1
f is conc. down  conc. up
```

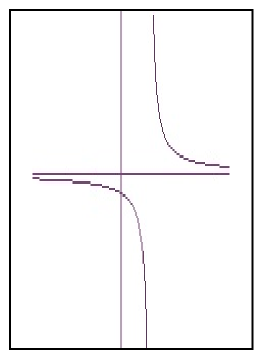

Figure III.D.2
Graph of $\frac{1}{x-1}$

Table 3

x	$\frac{1}{x-1}$
0.9	**-10**
0.99	**-100**
0.999	**-1000**
1.1	**10**
1.01	**100**
1	**1000**

Certainly $f''(x)$ is not defined at x=1 only, while for $x > 1$, f''(x) >0 and for $x < 1$, f''(x) < 0. The fact then that f is concave up on the interval (1,∞) supports the shape of the vertical asymptote with $f(x)$→∞ as x↓1 . Similarly that f is concave down on the interval (-∞,1) is consistent with $f(x)$→-∞ as x↑1.

Vertical tangent lines and Cusps. We have seen that a function that is not differentiable can still be continuous. One example was the absolute value function which has a graph that comes to an angular point at (0,0). Not only is this function not differentiable at 0 but its graph fails to have a line that might be described as tangent at (0,0). This situation is quite different for the function $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. See Figures 3 and 4. Although it is not immediately obvious from the transformation figure, when juxtaposed with the graph, the behavior of this function at 0 is quite different in some ways from most functions we have studied so far with the calculus. As $x \to 0$, $f(x) \to 0$, yet if we look at the graph of f we see that the Y axis appears to be a crossing tangent line. [This is similar to the way the X axis is a crossing tangent to the graph of the graph of $y=x^3$ at (0,0).] Looking at the definition of the derivative interpreted graphically, we examine the slopes of the secant lines between (0,0) and $(h, f(h))$. $m(h) = f(h)/h = h^{1/3}/h = h^{-2/3}$. So when $h \to 0$, $m(h) \to \infty$, and there is no slope that would be suitable for a tangent line. This is consistent with the fact that vertical line have no slopes.

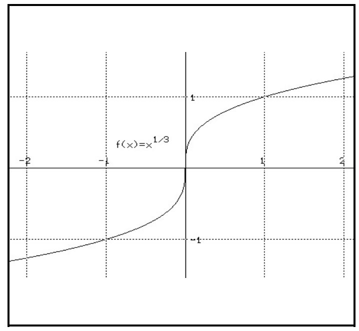

Figure 4

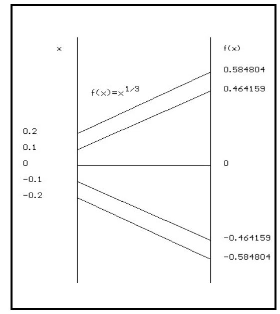

Figure 3

Table 4

h	$\sqrt[3]{h}$	$h^{\frac{-2}{3}}$
-0.1	-0.464	-4.6416
-0.01	-0.215	-21.544
-0.001	-0.1	-100
0.1	0.4642	4.6415888
0.01	0.2154	21.5443
0.001	0.1	100

REWRITE: In terms of the motion interpretation of the derivative, an object moving according to this rule would have an average velocity that would be getting larger and larger without bound as the object approached the 0 position at time 0. Strange as it may seem in a very short time, say .001 seconds, the object will have a relatively large distance of .1 meters to travel to arrive at 0, giving an average velocity of .1/.001 = 1000 meters/second= 1 kilometer per second, which seems quite speedy indeed! And the object will travel ever faster and faster as it approaches 0. Here is a situation where we can begin to question the meaning and usefullness of trying to measure the average velocity in a physical context.

Returning to the graphical interpretation we can make sense of the increasing slopes of secants without much difficulty. Not only are the secant slopes increasing without bound, but so are the slopes of the tangents at points along the curve as they are taken closer to (0,0). More importantly this is not difficult to recognize using the calculus. When

$x \neq 0$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, so $f'(x) > 0$ and as $x \to 0$, $f'(x) \to \infty$.

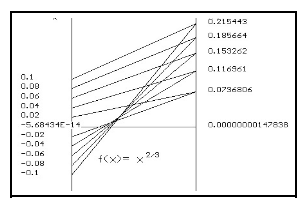

Figure 5

Another example of a vertical tangent line is $f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}}$. See Figures 5 and 6. In this example we have another vertical tangent at 0. Using the definition of the derivative interpreted as the slope of the tangent line estimated by secant line slopes is not as clear in this example since these slopes are large and negative when $x \uparrow 0$, but large and positive when $x \downarrow 0$. Since the magnitude of these slopes is getting large in either case, the

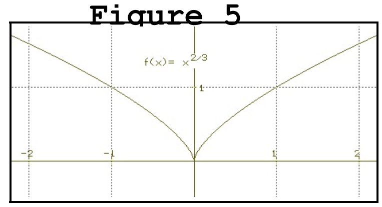

Figure 6

vertical tangent issue is resolved by noticing that $\left|\frac{h^{\frac{2}{3}}}{h}\right| = \left|h^{-\frac{1}{3}}\right| \to +\infty$ as $h \to 0$. Again this feature can be recognized using the derivative of the function. We analyze the magnitude of tangent lines slopes on the graph close to (0,0) with $|f'(x)|$. When $x \neq 0$, $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$, so as $x \to 0$, $|f'(x)| \to +\infty$. The first and second derivative analysis $[f''(x) = -\frac{2}{9}x^{-\frac{4}{3}}]$ also helps in understanding the shape of this section of the curve, usually **described as a *cusp***. Look at Figure 5 again.

```
f'(x)   --------T++++++++        T=undefined
        __|__|__|__|__|__|__
                0
f  is   decreasing  increasing
            ↘         ↗
f''(x)  ---------T--------       T=undefined
       _|__|__|__|__|__|__|__
                 0
f  is   conc. down  conc. down
```

Horizontal Asymptotes and Behavior at Infinity: What happens in the long run. It doesn't take much thought or experimentation to understand that when the number x is very large and positive, the value of 1/x is close to 0. In this very simple example are some key ideas for understanding the behavior of a function at infinity as well as horizontal asymptotes for the graphs of functions. The situation where x is very large and positive is sometimes written symbolically as $x >> 0$ (read "x is much greater than 0") or $x \to \infty$ (read " x approaches infinity"). The comparable case where x is very large and negative is treated similarly using $x<<0$ and $x\to-\infty$.

Now the question of what happens to the values of a function of x when x assumes very large values whether positive or negative is usually called the analysis of the **behavior at infinity**. Note that infinity in this usage is not a number, but refers only to the dynamic interpretation of larger and larger values of x being considered without bound. If the controlling variable is considered as time in an interpretation, then the question of how the controlled variable behaves can be thought of as the question of "what will happen in the long run" for large values of x and what happened "way back in time," before any history. For many functions there is no need to use calculus to investigate these questions, just as no calculus was needed to recognize that 1/x would be close to 0 when $x >>0$ or when $x<<0$. The analysis of the first and second derivatives can in some cases allow us to understand a little more about this type of behavior. so let's spend a little more time with $f(x) = 1/x$ for $x\neq0$ to see how this, and similar, examples can be enriched by further analysis. Noticce that when $x >0$, $1/x > 0$, while $f'(x) = -1/x^2 <0$ for all $x\neq0$. So we have the function f is decreasing for the intervals $(-\infty,0)$ and $(0,\infty)$.

A continuous decreasing function bounded below will have a horizontal asymptote.

As our simple example of 1/x shows, the behavior of a function at infinity may be asymptotic, meaning that as x gets large, the values of the function get close to some other quantity. In fact for many simpler examples the analysis to find that limiting quantity follow a rather simple thought process as the next example illustrates.

Exercises III.D.

For problems 1 - 14, find the indicated limits as most appropriate when they exist.

1. $\lim_{x \to -1} \frac{3x^2 + 4x + 1}{x + 1}$

2. $\lim_{x \to -1^+} \frac{x^2 + 4x + 1}{x + 1}$

3. $\lim_{x \to 1} \frac{3x^2 + 4x + 1}{3x + 1}$

4. $\lim_{x \to \infty} \frac{5x^2 + 4x + 1}{8x^2 + 10}$

5. $\lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1}$

6. $\lim_{x \to \infty} \frac{3x^2 + 4x + 1}{6x^3 + 1}$

7. $\lim_{x \to -1^-} \frac{x^2 + 4x + 1}{x + 1}$

8. $\lim_{x \to -\infty} \frac{3x^2 + 4x + 1}{6x^2 + 1}$

9. $\lim_{x \to -1} \frac{3x^2 + 4x + 1}{x^3 + 1}$

10. $\lim_{x \to \infty} \frac{3x^2 + 4x + 1}{6x + 1}$

11. $\lim_{x \to -1} \frac{3x^2 + 4x + 1}{x^4 + 1}$

12. $\lim_{x \to \infty} \frac{3x^3 + 4x + 1}{6x^3 + 2x + 8}$

13. $\lim_{x \to 1^+} \frac{3x^2 + x + 1}{x - 1}$

14. $\lim_{x \to -\infty} \frac{3x^3 + 4x + 1}{6x^4 + 1}$

15. Find $\lim_{x \to \infty} \sin\left(\frac{2\pi x^2 + 4x + 1}{8x^2 + 10} \right)$.

16. Discuss the asymptotes of the function $f(t) = \frac{(t^2 - 4)}{(t^2 - 9)}$

a) using the first derivative to analyze $f(t)$.

b) using the second derivative to analyze $f(t)$.

17. Discuss the asymptotes of the function $f(t) = \frac{(t - 4)}{(t^2 - 9)}$

a) using the first derivative to analyze $f(t)$.

b) using the second derivative to analyze $f(t)$.

18. A function F is called a **continuous probability distribution function over [a,∞)** if F is a continuous nondecreasing function and F(a)= 0 while $\lim_{t \to \infty} F(t) = 1$.

Using this definition show that the following are continuous probability distribution functions on [0,∞).

a. $F(t) = 1 - 1/(t+1)$.   b. $\sin(\pi/2 \; t/(t+1))$.

19. a) Explain why if f is an even function and $\lim_{t\to\infty} f(t) = L$ then

$\lim_{t\to -\infty} f(t) = L$.

b) Suppose f is an odd function and $\lim_{t\to\infty} f(t) = L$. Show $\lim_{t\to\infty} f(t) = -L$.

20. We say that f has g as an asymptote if $\lim_{t\to\infty} [f(t) - g(t)] = 0$.

Show that $f(x) = [x^2 + 1] / x$ has $g(x) = x$ as an asymptote.

21. Draw a transformation figure and a graph to explain why

$\lim_{x\to\infty} f(x) = 0$ if and only if $\lim_{x\to\infty} |f(x)| = 0$.