Chapter III.C Analysis of The Second Derivative
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In this section we will develop applications of the second derivative that explain more graphical features of functions. These features are described geometrically by the terms *concavity* or in some contexts *convexity*. Using the motion interpretation the second derivative can be understood as acceleration. The second derivative also provides another tool for the study of extrema. It gives a quick, though not always effective, test for local extremes at critical points.

As we proceed through this section, keep in mind the interpretation of the second derivative as the rate of change of the first derivative values.

Position: The second derivative may be interpreted as the rate of change of velocity (*i.e.*, acceleration) in the motion interpretation.	Graph: The second derivative may be interpreted graphically as the rate of change of the tangent line slopes.
Economics: The second derivative of production related measurements like costs, revenues, and profits, gives the rate of change of the corresponding marginal rates as production levels change.	Probability: The second derivative of the distribution function determined by a continuous random variable gives the rate of change of the point density function values as the value of the random variable changes.

The second derivative, acceleration, mapping figures, and graphs. We will use the motion interpretation of a function to analyze the function's second derivative in relation to its graph and mapping figure.

The Baton Pass: Consider two runners, Fred and Ginger, in a relay race on a straight coordinate line about to pass a baton. Fred is the runner who has the baton and Ginger will receive the baton. [Insert Photo of Fred and Ginger passing baton]

Let $f(t)$ denote Fred's location at time t using the function f, while $g(t)$ will denote Ginger's location at the same time. To pass the baton smoothly, Fred and Ginger want to be at the same position and traveling at the same velocity at time $t = a$.

So, for a smooth baton pass at time a, Fred and Ginger want $f(a) = g(a)$ and $f\,'(a) = g\,'(a)$.

After the baton is passed, Ginger will accelerate, *i.e.,* $g''(t) > 0$ for $t > a$, while Fred will continue to run at the same velocity, *i.e.,* $f\,'(t) = f\,'(a)$ for $t > a$. It should make sense that Ginger's coordinate will increase faster than Fred's after the pass, so that for $t > a$, $g(t) > f(t)$.

Representing Fred and Ginger's positions in mapping Figure 1 illustrates their relative situations at times *b, c* and *d* after $t = a$.

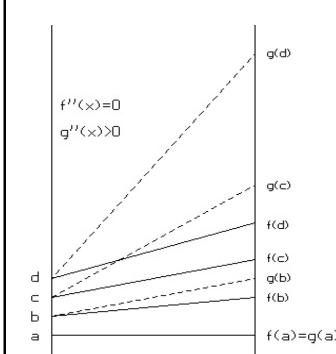


Figure 1

Looking at the graphs for the same scenario, we see the two function as curves. Since Fred's velocity is constant for $t > a$, the graph of $f(t)$ for $t > a$ is in fact a line. Ginger is accelerating, *i.e.,* $g''(t) > 0$ for $t > a$, so the graph of g is above the graph of f for $t > a$. Because $f(a) = g(a)$ and $f'(a) = g'(a)$, the graph of f is the tangent to the graph of g at $(a, g(a))$. See Figure 2.

Conclusion: For $t > a$, the graph of the function g lies above the line tangent to the graph of g at the point $(a, g(a))$.

Similar reasoning can be used to see that when $g''(t) > 0$ for $t < a$, the graph of $g(x)$ is above the line tangent to the graph of g at $(a, g(a))$. Thus, if $g''(t) > 0$ for all t in an interval containing a, the graph of g lies above the tangent line at $(a, g(a))$.

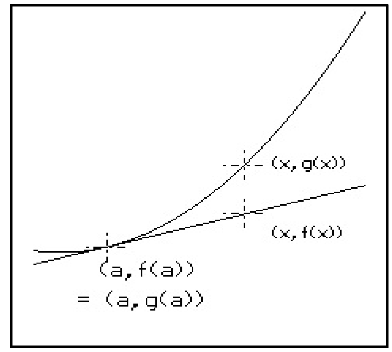


Figure 2

We'll continue this discussion after we introduce some of the key ideas related to the geometric concept of concavity.

START HERE **Geometry, concavity, and convexity.** The words **concavity** and **convexity** have a long history of use in science as well as in mathematics – going back as far as Archimedes (ca. 287 - ca. 212 BCE) – to describe shapes of curves, planar regions, surfaces, and solids. There are many ways to characterize these features, two of which we will illustrate before adopting one as the definition for our discussion.

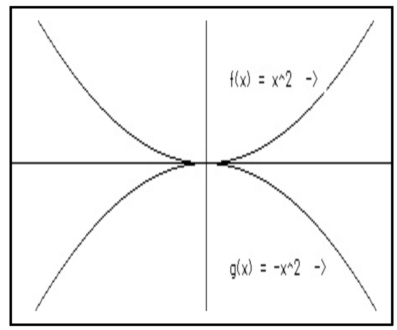
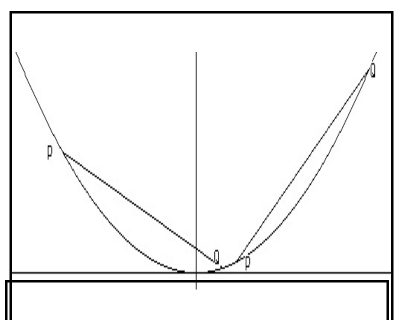


Figure 3

Let's use C to denote the graph of $f(x)= x^2$ and D for $g(x) = -x^2$. See Figure 3. To avoid confusion on the terminology between concave and convex, we will describe shapes similar to C as concave up and those similar to D as concave down.

Concave Up. 1. Choose any two points P and Q on C and draw the line segment PQ between them. After doing this several times our first characterization for curves that are concave up should make sense. For a curve that is **concave up, a line segment between any two points on the curve will lie above the curve.** See Figure 4. This gives a simple **test** to show that a curve **is not concave up**. You need only **find a line segment between two points on the curve that does not lie above the curve** (as in Figure 5). It is more difficult to show this property **is satisfied** for a curve because it must be satisfied by every line segment between any pairs of points on the curve.



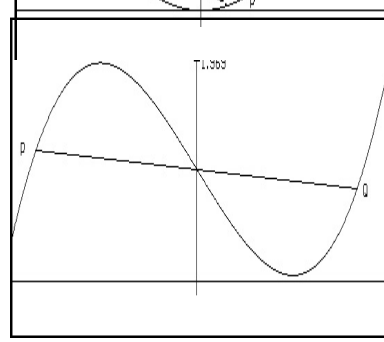


Figure 5

Concave Down. 1. Treating the curve D to the same analysis that C received gives a comparable description of concave down. Thus any **line segment between two points on a curve that is concave down will lie below the curve. See Figure 6.**

Concave Up. 2. A second way to characterize curves that are concave up uses tangent lines. Draw several tangent lines to C. See Figure 7. Each of these lines lies below C. This then is the second geometric quality that characterizes a curve that is concave up, -namely: in a vicinity of any point **the tangent line to the curve at the point lies below the curve.**

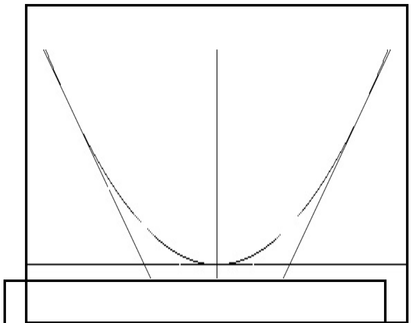


Figure 7

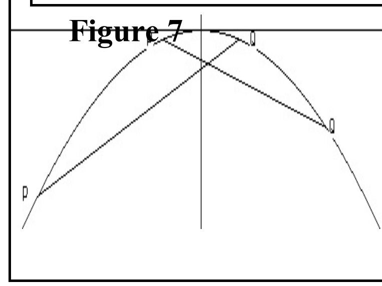


Figure 6

NOTES- A. The characterization using the tangent line allows us to consider a curve concave up at one point. Testing for this property at a point is organized easily with the geometry. You need to find the tangent line at the point and then recognize its position relative to the curve. This is precisely the information we noticed when we analyzed Fred and Ginger's motion.

B. Having drawn several tangents to the curve C brings to light another aspect of concavity . As the points progress from left to right on the curve the slopes of the tangent lines get larger. This rather simple observation turns out to provide another key for a **practical test for concavity: Examine the first derivative of a function defining the curve to see on what intervals the derivative is increasing.**

Since the geometrical interpretation of the derivative is that it gives the slope of the tangent line, an increasing derivative can be interpreted as increasing slopes for the tangent lines. One way to discover how the first derivative is behaving is to **do first derivative analysis** on the **first derivative. So if the derivative of the first derivative, *i.e.* the second derivative, is positive on the interval, then the first derivative values are increasing on the interval.**

> From Archimedes, On the Sphere and Cylinder,
>
> DEFINITIONS.
>
> 1. There are in a plane certain terminated bent lines, which either lie wholly on the same side of the straight lines joining their extremities, or have no part of them on the other side.
>
> 2. I apply the term **concave** in the same direction to a line such that, if any two points on it are taken, either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself, but none on the other side.

Concave Down. 2. Treating the curve D to the same analysis that C received gives a comparable description of concave down using the tangent lines. Thus the **tangent lines for a curve that is concave down for an interval will lie above the curve . Furthermore in this case the tangent line slopes will be decreasing, which is guaranteed if the second derivative is negative for the interval.**

We now give a geometric definition of concavity which will allow us to formulate in more precise language the geometric results of our analysis of the second derivative.

Definition: We say that the graph of a function f is **concave up over an interval I** if for any a and b in I, and any x where x is between a and b,

$$f(x) \le f(a) + \frac{f(b)-f(a)}{b-a}(x-a).$$

Concave UP- The function's graph is below the secant line.

We say that the graph of a function is **concave down over an interval I** if for any a and b in I, and any x where x is between a and b,

$$f(x) \ge f(a) + \frac{f(b)-f(a)}{b-a}(x-a).$$

Concave UP- The function's graph is below the secant line.

Though these definitions of concavity do not involve the concepts of tangent lines (or derivatives), there is a connection to the calculus which we have already noted and which the following results make explicit. The proofs of these results are contained in Appendix III.***

Theorem III.C.1: [A monotonic derivative is equivalent to Concave.] Suppose f is a differentiable function on an open interval I .(P) f is **concave up** on I if and only if for any a and b in I where a < b, **f'(a) ≤ f'(b).** (N) f is **concave down** on I if and only if for any a and b in I where a < b, **f'(a) ≥ f'(b).**

Proof: See Appendix III.*.**

Theorem III.C.2:[Tangent Lines and Concavity] Suppose g is differentiable on an open interval I.

(P) g is concave up if and only if for any point c in I and any point x in I,

$$g(x) \ge g(c) + (x-c)\, g'(c).$$

(N) f is concave down if and only if for any point c in I and any point x in I,

$$g(x) \le g(c) + (x-c)\, g'(c).$$

Proof: See Appendix III.***.

Note: Concavity, The Differential, and Estimations. The expression **g(c) + (x-c) g'(c)** is in fact the differential estimate for g(x) at c using dx=x-c, i.e., dg(c,x-c)=(x-c) g'(c) so Theorem III.C.2 says that if g is concave up at c then differential estimate of g(x) at c, namely, g(c) +dg(x,x-c) will be an underestimate of the exact value of g(x). This is consistent with motion interpretation thinking of Fred's position as the linear-differential estimating function while Ginger's position can be considered as the function g which is increasing at increasing rates.

With these two theorems stated the analysis of concavity becomes relatively easy using the second derivative as the next result shows.

Theorem III.C.3: Suppose f is a continuous function considered on an interval I.

(P) If f''(x)>0 for all x inside I then f is concave up over I.

(N) If f''(x) < 0 for all x inside I then f is concave down over I.

Proof: (P) We suppose f''(x) > 0 for x inside I, so f'(x) is increasing on the interval I, i.e., if a and b are in I and a<b then f'(a) < f'(b). Now Theorem III.C.2 can be applied and f is concave up over I.

(N) This is done similarly and is left as an exercise for the reader.

Points of inflection: A point on the graph of the function which is the boundary between intervals where the function's graph has different concavity is called **a point of inflection.** See Figure III.C. ***. You can use the second derivative to discover and confirm points of inflection of a function f when f'' is a continuous function. Find those points where f''(x) = 0. [These can be described as the second order critical points for f in analogy with those points where f'(x)=0.] By the intermediate value theorem applied to f'', these are the only points where f''(x) can change from positive to negative or vice versa. Thus these are the only points where there can be a change in concavity. Now check to see if there is a change in the sign of f''(x) by testing points to the left and right of these points. If there is a change in sign, then you have found the first coordinate of a point of inflection.

Tangent lines and inflection. On one side of a point of inflection the analysis for positive second derivative will apply while on the other side the analysis for negative second derivative will apply. Therefore on one side of a point of inflection the tangent line will lie above the curve while on the other side the tangent line will lie below the curve. So at a point of inflection the tangent line will cross the curve. This is one of the most visible features of this change in concavity at the point of inflection. See Figure III.C*?

Interpretations of points of inflection: 1. Consider f as the position function for a moving object. We can then interpret f'' as the acceleration of the object. A point c where f''(c) = 0 is a time where the acceleration is 0, and thus is a critical point for f', the velocity. First derivative analysis shows that if there is a change in sign of f'' at c, then we have a local extreme value for the velocity at time c. Therefore a point of inflection can be interpreted as a point in time where the velocity achieves a relative extreme value.

2. Suppose F is a probability distribution function for a continuous random variable X. If (c,F(c)) is a point of inflection for the graph of F, then c is an extreme value for F'(x) = f(x) , the probability density function of X. Thus c is a possible mode for the random variable X.

Example III.C.1 Suppose $f(x) = x^4 - 6x^2 + 2$. a) Find those intervals where the graph of f is concave up and concave down.

b) Find any points of inflection.

Solution: a) $f'(x) = 4x^3 - 12x$. $f''(x) = 12x^2 - 12$. Solving for when f''(x) = 0 we have $0=12x^2-12$ so $x = \pm 1$. Using the continuity of f'', we can see that when x > 1 or when x<-1, f''(x)> 0, while when -1< x < 1 , f''(x) < 0. Therefore by applying the theorem f is concave up on the intervals $(1,\infty)$ and $(-\infty,-1)$, and concave down on the interval (-1,1).

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f''(x)    ++++++++0----------0++++++++

          _|___|___|___|___|___|___|___|___|___|_

          -2    -1     0     1     2
f' is        incr.  decreasing    incr.
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Figure 8

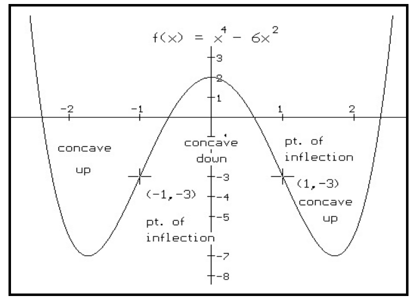

Figure 9

x	$y=f(x)=x^4-6x^2+2$	$\Delta y=f(x+.5)-f(x)$	$f'(x)$	$f''(x)$
3	29	25.4375	72	96
2.5	3.563	9.5625	32.5	63
2	-6	0.4375	8	36
1.5	-6.4375	-3.4375	-4.5	15
1	-3	-3.5625	-8	0
0.5	0.5625	-1.4375	-5.5	-9
0	2	1.4375	0	-12
-0.5	0.5625	3.5625	5.5	-9
-1	-3	3.4375	8	0
-1.5	-6.4375	-0.4375	4.5	15
-2	-6	??	-8	36

b) From part a it is clear that at x=1 and x=-1 there is a change in the sign of f" and thus a change in concavity. Since f(1)= f(-1) = -3 we have (-1,-3) and (1,-3) are points of inflection for f.

Comment: Though the graphical approach to interpreting the second derivative is very valuable, it is worth noting that the second derivative can also be used to understand the relative size in the differences between the function's values. As Table *** demonstrates, the differences between values of f appear to be locally smallest at x = 1 and locally largest at x=1. These function value differences when divided by the change in x can be used to approximate the derivative. So we should not be too surprised to find that the values of the derivative in the same table show a similar feature. The second derivative's values graphed in Figure *** also present information consistent with the graph of *f* '(x) as seen in Figure ***. That is, when *f* "(x)>0, the function f' is increasing, and when *f* "(x)<0 then the function *f* ' is decreasing.

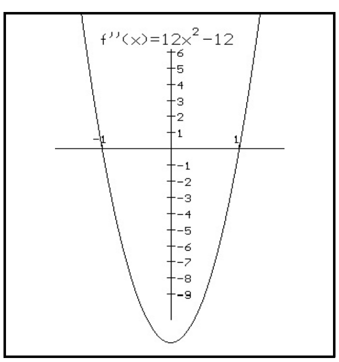

Figure 10

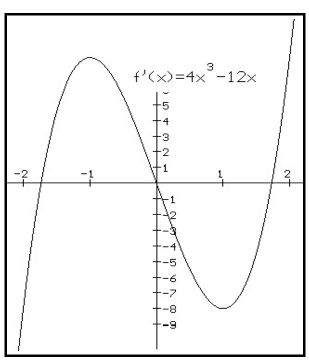

Figure 11