CHAPTER III Applications of the Derivative (Draft- work in progress)

Introduction: As we saw in Chapters I and II, the derivative can be a powerful analytic tool for any quantifiable scientific discipline. In many interpretations this concept consolidates information about changing relations between variables.

The symbolic and numeric methods for finding or estimating the derivative as a number or as a function make it efficient, eliminate repetitive arguments and computations, and allow the focus of its use to remain on the applications, keeping most mathematical manipulations routine and secondary.

The visual tools of graphs and transformation figures provide additional power to the concept by supplying alternative meanings for more abstract contexts. Applications of the derivative are not limited to those previewed in chapter I or to those that we will discuss in this chapter. In fact one could easily describe everything in this text as an application of the derivative in some fashion!

We will analyze three types of applications in this chapter. Each has a wide scope of impact extending well beyond the specific context in which we will see them initially.

A. **Estimations:** Understanding estimations is a critical part of any science that uses measurements. **The key** in using the derivative to estimate values lies in the assumption that a function with **a derivative behaves like a function with a constant rate of change for some small interval.** In this chapter we will use the derivative in two common estimation procedures. First, we will **estimate the value of a function** thinking of it as a position function in a motion interpretation. Then we will discuss an algorithmic process called Newton's method to **estimate a zero of a function or root of an equation** thinking of this number as a time when a position function is at the initial distance or graphically as the X-intercept coordinate of a curve.

B. **Graphs:** The important role played by graphs in current science is undeniable. It is the chief tool of visualization of data and provides an efficient vehicle for suggesting relations between variables. With the power that technology adds to the creation and display of graphs, the need for **better understanding of graphical features and their analysis** has risen while the need to **analyze functions represented symbolically for quick and accurate drawings** has diminished only slightly. As we saw in Chapter I, an understanding of the derivative can bring with it a richer appreciation for graphic function properties.

C. **Modeling:** It is in the sciences that the use of the derivative as a mathematical concept pays for the time and effort it takes to master its notation and rules. Without the ability to bring this abstraction into more then just the problems generated by mathematics alone, it would be hard to see why mathematics is as prominent as the "language of science." Even with the rise of computational power through technology, the use of the derivative continues as before **to describe and investigate the world through the measurement of variables**.

In the next chapter we will look more extensively at the ramifications of describing a context by relating the rates at which variables change.

In this chapter we will look at models where **the relation between the variables can be expressed in some equation or with a function with one controlling variable.** Though this may seem limited in scope, the questions we will examine for these models are of sufficient generality to make them good **prototypes for more general models**.

The questions are simply **how to use the information available to predict** the behavior of variables under specific constraints. Just as with the graphical applications, the derivative can inform us of extremes, intervals where variables will increase and decrease, even the rates at which the rates of change change.

III. A. ESTIMATIONS USING THE DERIVATIVE

III.A.1. THE DIFFERENTIAL

MOTIVATION: Consider a jogger running on a straight track so that **after 2 seconds the jogger is 10 meters from the starting point P and at that moment is moving away from P with a velocity of 3 meters/sec**. I would like to **approximate the position of the jogger 0.4 seconds later.**

It seems reasonable to assume for this estimation that the velocity won't change much in .4 of a second. So we **treat the velocity of the jogger as a constant**.

Now it should be apparent that **in 0.4 of a second the jogger will move approximately (0.4)(3) = 1.2 meters** further away from P. So in 0.4 of a second the jogger will be approximately 10 + 1.2 = 11.2 meters from P.

We can express this analysis more technically using some function notation. Let ***t*** denote the time in seconds and s(***t***) denote the jogger's distance from P at time ***t***. Then the initial facts were that **s(2) = 10** and that **s'(2)=3**.

To estimate the change in the value of s for 0.4 seconds we multiplied s'(2), the rate at which the jogger was running, by 0.4, the time the jogger would be running.

In symbolic form we had **$s(2.4) - s(2) \approx s'(2)\ 0.4 = 3\ (.4) = 1.2$.**
Now we complete the analysis by adding the estimate of the change to the runners position at 2 seconds giving $s(2.4) = s(2) + \{s(2.4) - s(2)\} \approx 10 + 1.2 = 11.2$.

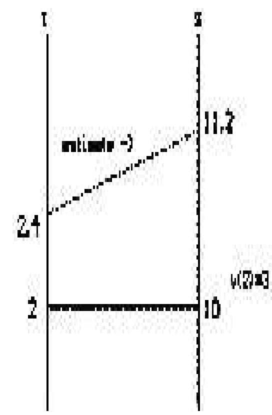

Figure 1

The simple technique we used here generalizes to a method for **estimating values of any differentiable function based on information about the value of the function and its derivative at a single point.** The key is using the product of the value of the derivative with a small change in the controlling variable to estimate the change in the corresponding change in the function's value.

The word that has been used since Leibniz to describe the estimate for the change is the "differential."

Definition and Notation: Suppose *f* is a function that is differentiable at *a* and h is any real number. We'll write d*f* = *df* (*a*,h) = *f* '(*a*).h .

d*f* is called "the differential of *f* at *a*."

As the more complete notation d*f* (*a*,h) suggests, the quantity *df* depends on both the numbers *a* and h. When the notation *y* = *f* (*x*) is used to describe the function, then we also denote the differential with d*y* as well as d*f* .

It is important to distinguish *df* from *Df.*
The notations are very close so keep in mind:

* d*f* is **the differential of *f*** which gives an estimated change in the value of the function *f*,
* D*f* is **the derivative of *f*** which gives the rate of change of *f*.

REMARK: We can apply this new notation to the motivating example to focus its use more sharply. Recall that s is the position function for the jogger which depends on the time variable ***t***.

The **differential of s at 2** is therefore given by **ds = ds(2,h) = s'(2) h ≈ s(2+h)-s(2)**.
Now using s(2) = 10 and s'(2) = 3, we have s(2 + h) ≈ s(2) + ds = 10 + 3h for any h.
So when h = 0.4 we have s(2.4) ≈ 11.2.

Of course, in the jogger situation, the closeness of this estimate depends on how the runner's velocity, s'(t), actually varies between 2 and 2.4 seconds. It should seem reasonable to suppose that since 0.4 is close to 0, the estimate is fairly accurate and that it would be more accurate for choices of h even closer to 0.

Interpretation (motion): We can interpret the number *a* as a time at which we know information about the value of position function, *f*. The number **h** can be thought of as **measuring a time interval** we add to (or subtract from) *a* to determine a later (or earlier) time at which we would like to know the function's value.
The value of d *f* is then an estimate of the change in position, the function's value, from time *a* to this later (or earlier) time, *a*+h. Thus **d *f*(*a*,h) ≈ *f* (*a*+h) - *f* (*a*).**

Interpretation (Transformation figure): Assume h >0. In the transformation figure below [Figure 2] we have labeled the lengths of the key elements determining the differential, namely the points on the source line, *a* and *a*+h, and the points on the target line: *f* (*a*), *f* (*a*+h), and the estimate based on the differential, *f* (*a*) + *f* '(*a*) h. When h is small we have that the average rate of change of the f values on the interval [*a*, *a*+h] is a good estimate for the instantaneous rate of change of *f* at *a*, *f* '(*a*). That is,

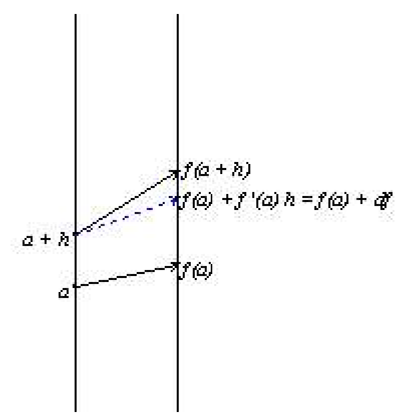

Figure 2

$$\frac{f(a+h)-f(a)}{h} = \frac{\Delta y}{\Delta x} \approx f'(a)$$

.

Thus $\Delta y = f(a+h) - f(a) \approx f'(a) \bullet h = df$.

Looking at the target line we see the estimating value compared to the precise value of *f* (a+h). For the estimate, the length *f* (a) is increased (decreased) by the length d*f*. When h is closer to 0, the differential's length will be closer to the length of the segment between *f* (*a*+h) and *f* (*a*), so the estimation will be more accurate.

Interpretation (The graph and the tangent line): The differential of *f* at *a* can also be visualized using the interpretation of the derivative as the slope of the line tangent to the graph of *y* = *f* (*x*) at the point (*a*, *f* (*a*)). Figure 3 shows the lengths of the key elements used in determining the differential, namely the points on the graph of *f*, (*a*,*f* (*a*)) and (*a*+h,*f* (*a*+h)), and on the tangent line (*a*+h, *f* (*a*)+ *f* '(*a*) h).

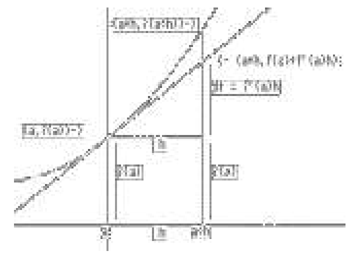

Figure 3

Note that the second coordinate of the point on the tangent line was determined from the fact that the slope of the tangent line is *f* '(*a*). When h is small we have that the slope of the secant line determined by (*a*,f(*a*)) and (*a* + h,*f* (*a*+h)) is a good estimate for the slope of the tangent line, *f* '(*a*), that is,

$$\frac{f(a+h) - f(a)}{h} = \frac{\Delta y}{\Delta x} \approx f'(a)$$

.

By multiplying both sides of this estimate by h we find yet another way to understand the estimate $\Delta y = f(a+h) - f(a) \approx f'(a) \bullet h = df$.

We can examine the figure in greater detail to see how the differential is used to estimate *f* (*a*+h). The vertical line segment used to measure *f* (*a*+h)-*f* (*a*) = Δy can be compared to the vertical segment of the right triangle formed by the tangent line, the vertical line X = *a*+h and the horizontal line Y = *f* (*a*). The base of this triangle has length h = Δx = d*x*. Because the tangent line has slope *f* '(*a*) the segment on the vertical line X = *a*+h has must have length *f* '(*a*)h = d*y*. Again we have a way to see that d *y*/h = d*y*/d*x* = *f* '(*a*), the derivative is the quotient of differentials! The estimation can be seen by thinking of the point (*a*+h, *f* (*a*+h)) approximated by extending the vertical segment of the line X = *a*+h from the point (*a*+h, *f* (*a*)) by the length d *y*.

Interpretation: (Economics): Consider a function model for the cost C of producing *x* kilograms of a perfectly divisible commodity. As we saw in Chapter I, when we produce *a* kilograms, the marginal cost is C '(*a*). If we decide to produce an additional h kilograms of our product, we can estimate the change in our costs and our new costs with the differential at *a*. Thus ΔC ≈ dC= dC(*a*,h) = C '(*a*) h and C(*a*+h)≈C(*a*) + dC = C(*a*) + C'(*a*) h. [The marginal cost at *a*, which we denoted MC(*a*) in chapter I, originally meant the change in the cost for a change of one unit of production. Now the MC(*a*) can be connected more directly to the derivative by using the differential estimation, giving MC(*a*) ≈ dC(*a*, 1)= C '(*a*).] See exercises.....

Interpretation: (Probability). Consider a continuous random variable X with a distribution function F defined on an interval I. The probability that X is between the values A and A+h, Pr(A<X<A+h), is given precisely by F(A+h) - F(A), the change in the distribution function values for the interval [A, A+h]. Using the differential to estimate the change in the distribution function values, we have that this probability is estimated by dF(A,h)=F '(A) h. Now F '(A) is the point probability density of X at the number A, which is conventionally denoted by *f* (A). Thus for small intervals, that is, when h is small, the probability that the random variable X is between A and A+h is approximately *f* '(A) h, the value of the density function for X at A times the length of the interval. See exercises....

More on notation: The notation of the differential was introduced by Leibniz with a different view of what it represented from its current use. In particular Leibniz used the symbols d*y* and d*x* to represent the measures of very small- I mean extremely small- segments measuring the rise and run of very short sections of a curve. Thus it appears that Leibniz was interested in finding the slope of a curve by inspecting the curve very closely. This is analogous to what we can do today with graphical technology zooming in so that the graph of a curve appears to be indistinguishable from the graph of a line.

We continue in this section to develop the notation for differentials, allowing us to make some sense out of this older view. The Leibniz notation has proven particularly successful in connecting many concepts to scientific applications and so continues to be used widely in the sciences.

What does d*x* denote? Suppose $x(t) = t$ for all t. Then $d\,x = d\,x(a,h) = x'(a)\,h$. But $x'(t) = 1$ for all t, so $d\,x = h$. This bizarre consequence of the notation justifies the **abuse of notation** in saying that when $y = f(x)$,

$$\mathbf{d\,y = d\,f = f\,'(a)\;\; h = f\,'(a)\; d\,x.}$$

It is possible now to make sense in many situations of Leibniz notation. Even though the original use of this notation most likely had a very different though consistent meaning to Leibniz and others historically, we can now give the notation of **d*y*/d*x* an interpretation as a quotient!**.

$$\mathbf{dy/dx = d\,f\,/d\,x = f\,'(a)\; d\,x/\; dx = f\,'(a)}.$$

Another comment on notation. In most American mathematics books the notation for the change in a variable's value uses the Greek letter, upper case delta, Δ, such as $\Delta\, x$ for change in the variable x and Δt for change in the variable t. In some physics books and British mathematics books the lower case of the same letter, δ, so you might see in δx or δt. For some users of these notation a minor distinction might be made in the use. Δx would be used for any change in the variable x, whereas dx or δx would indicate that there was a very small change in the variable x, so small that it would be almost imperceptible.

More Notation: In Chapter I we used h for some of our initial derivative estimations. We suppose again that y = $f(x)$ and let

$$\Delta\, x = h \;\text{ and }\; \Delta\, y = \Delta y(a,h) = f(a+h) - f(a).$$

By our previous comments then

$$dx = \Delta x = h \text{ and } \Delta y = f(a+dx) - f(a).$$

Estimates with the differential: The heart of the matter in making an estimate of $f(a+h)$ with the differential is the fact that for small values of h, $dy \approx \Delta y$, so that

$$\boldsymbol{f(a + dx) = f(a)+\Delta y \approx f(a) + dy = f(a) + f'(a)\, dx.}$$

EXAMPLE: (Let's try it.) Find dy when $y = f(x) = x^3 - 5x + 7$.
Evaluate dy when $x = 2$ and $dx = .3$. Use dy to estimate y when $x = 2.3$.
Find $f(2.3)$ and Δy exactly.

SOLUTION: Using $y = f(x)$, we have that $f'(x) = 3x^2 - 5$ so

$$dy = f'(a)dx = (3a^2 - 5)\, dx.$$

To evaluate dy we merely replace a with 2 and dx with .3 in the expression to obtain
$dy = (3(2)^2 - 5)(.3) = 2.1$

Noticing that $f(2) = (8 - 10 + 7) = 5$, we estimate

$f(2.3) = f(2 + .3) \approx f(2) + dy = 5 + 2.1 = 7.1$.

It is not hard to find $f(2.3) = (2.3)^3 - 5(2.3) + 7 = 7.667$, so that $\Delta y = 7.667 - 5 = 2.667$.

It is worth noting here that the size of the error in the differential estimate of $f(2.3)$ is the difference $7.667-7.1 = f(2.3) - (f(2) + dy) = \Delta y - dy = 2.667 - 2.1 = .567$.
We can compare this error in estimating the difference in values as a percent of the size of the precise difference Δy to find the percent of relative error in this estimate is $.567/2.667 = .2126 = 21.26\%$.

EXAMPLES: To see the relative quality of the estimate for values of a function using the differential let's look at the sine function values using the differential at 0. We compare these in Table 1 with the estimation values that arise from the differential. Since sin(0)=0 and sin'(0)= cos(0)=1, the estimate for sin(0+h) is sin(0)+sin'(0)A h=h. So Table 1 demonstrates that when $h \approx 0$, $\sin(h) \approx h$.

[This should remind you of the fact we demonstrated in chapter I, namely that when $h \to 0$, $\sin(h)/h \to 1$.]

The errors in the estimates of this table are clearly smaller when h is closer to 0. [Can you see why we haven't considered relative errors here?]

h	sin(*h*)	estimate
0.1	9.9833416647e-02	0.1
0.01	9.9998333342e-03	0.01
0.001	9.9999983333e-04	0.001
0.0001	9.9999999833e-05	0.0001
0.00001	9.9999999998e-06	0.00001
0.000001	1.0000000000e-06	0.000001

A similar comparison for $f(x)=1/x$ using the differential at 1 shows a less symmetric situation. Here $f(1)= 1$ and $f'(1)= -1$ so $f(1+h)$ is approximated by $f(1) + f'(1)h = 1 - h$. Table 2 shows how these estimates compare with some function values close to 1. Again the errors in the estimates are smaller when h is closer to 0, as are the relative errors which are shown as percentages in the table.

h	*a* =1+*h*	1/*a*	estimate	
0.5	1.5	0.6666667	0.5	33.33%
0.1	1.1	0.9090909	0.9	9.09%
0.01	1.01	0.990099	0.99	0.99%
0.001	1.001	0.999001	0.999	0.10%
-0.5	0.5	2	1.5	-100.00%
-0.1	0.9	1.1111111	1.1	-11.11%
-0.01	0.99	1.010101	1.01	-1.01%
-0.001	0.999	1.001001	1.001	-0.10%

Application: A hollow spherical steel ball has an inside radius of 2 meters and a thickness of 3 centimeters. Estimate the volume and the mass of the steel in the wall of the ball. Discuss the relative size of the error in using the differential to make this estimate.[Assume the density of the steel is 1254 kilograms per cubic meter.]

Solution: Let V(r) denote the volume of a sphere of radius r. This problem asks for an estimate of the volume of the steel in the wall of a sphere, which can be expressed as $V(2.03) - V(2) = \Delta V$. The formula for the volume of a sphere says that $V(r)= 4/3\ r^3$. Using a = 2 and h = .03 and the differential we have

$$\Delta V \approx dV = V'(a) .h = 4 a^2 .h = 4 * 4(0.03) = 0 .48 \quad \text{cubic meters}$$

and the mass of this volume is approximately 0.48 * 1264 = 606.72 kg

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Application: Use the differential to estimate $9^{1/3}$ from the fact that $8^{1/3} = 2$.

Solution: Consider $f(x) = x^{1/3}$. $f(9) = f(8+1)$ so in the notation we've established we let $a = 8$ and h=1. Thus $f(a+h) \approx f(a) + f'(a) \cdot h = f(8) + f'(8) \cdot 1 = 2 + (1/3)(8)^{-2/3}$

so $f(9) \approx 2 + 1/12 = 25/12 \approx 2.083333$.
[Check with your calculator that $9^{1/3} = f(9) \approx 2.0800838230519041145300568243579$]

Notice that what made this solution possible was the ability to evaluate both $f(8)$ and $f'(8)$. This ease of computation was what actually led to the choice of $a = 8$.

Application: Use the differential to estimate $98^{1/2}$.

Solution: Consider $f(x) = x^{½}$. f(98) = f(100 - 2) so in the notation we've established we let a=100 and h=-2. Thus

$$f(a+h) .f(a) + f'(a) . h = f(100) + f'(100) .(-2) = 10 - (100)^{-1/2}$$

and $f(98) \approx 10 - 1/10 = 9.9$.

Application: Error based on measurents. In measuring a cubical box the side was measured to have length of 2 meters with a possible error of .3 centimeters made in the measurement. Estimate the volume of the box. Discuss the possible error in the volume estimate based on the error in measuring the side. Estimate the relative size of the error in the volume estimate based on using the differential to make this estimate.

Solution: Let V(s) denote the volume of a cube with side of length s. With as side of 2 meters the volume is easily computed to by $V(s)= s^3$, so V(2) = 8 cubic meters. This problem asks for an estimate of the error can be expressed as $V(2.003) - V(2) = \Delta V$. Using $a = 2$ and h = .003 and the differential we have

$\Delta V \approx dV = V'(a) .h = 4a^2 .h = 4 * 4(0.003) = 0 .048$ cubic meters. The relative size of the error is the ratio of the possible error in the volume to the size of the computed volume, i.e.,

relative error = $\Delta V/V \approx dV/V = .048 / 8 = .006 = 0.6$ %.

The calculus of differentials.

Since the differential of a function is directly related to the derivative of the function, we can write formulas for a calculus of differentials each of which can be justified by reference to the appropriate derivative rule. For example, if u and v are both functions of x, then

d(u.v) = udv + vdu.

This is justified by considering the derivative product rule $D_x(u.v) = u D_x(v) + v D_x(u)$.

Hence $d(u .v) = D_x(u.v)dx = [u D_x(v) + v D_x(u)]dx = u D_x(v)dx + v D_x(u)dx$

$= udv + vdu.$

In the exercises for this section you are asked to justify similar results for the "differential calculus".

Exercises III. A:

For each of the functions in problems 1- 6 find (a) $f(1)$, (b) $f(1.2)$, (c) $dy(1,.2)$, and (d) $df(1,.2)$.

1. $f(x) = x^2 + 3x$
2. $f(x) = 5x^2 + 3x$
3. $f(x) = x^3 + 3x$
4. $f(x) = 5x^3 + 3x$
5. $f(x) = x^3 + x$
6. $f(x) = 5x^3 + x$

For each of the functions in problems 7-12 use the differential to estimate the value of (a) $f(1.1)$ and (b) $f(.95)$.

7. $f(x) = x^2 + 3x$
8. $f(x) = 5x^2 + 3x$
9. $f(x) = x^3 + 3x$
10. $f(x) = 5x^3 + 3x$
11. $f(x) = x^3 + x$
12. $f(x) = 5x^3 + x$

In problems 13-20, use the differential to estimate the indicated value.

13. $(82)^{1/2}$
14. $(63)^{1/2}$
15. $(127)^{1/3}$
16. $(25)^{1/3}$
17. 1/103
18. 1/998
19. $(33)^{1/5}$
20. $(29)^{1/5}$

21. Use the differential to give a formula for estimating $x^{1/2}$

when x is close to (a) 100 (b) 25 (c) 81 and (d) t.

22. Use the differential to give a formula for estimating $x^{1/3}$

when x is close to (a) 1000 (b) 125 (c) 27 and (d) t.

23. A circle of radius 3 meters is painted red with a edge of 10 centimeters painted blue. Use the differential to estimate the area of the region that is painted blue.

24. A spherical ball of radius 20 centimeters is coated with a shell of plastic .5 cm in thickness. Estimate the volume of plastic of the plastic shell.

25. A closed cylindrical tin can has radius 4 cm. and height 6 cm. Estimate the volume of the tin if the tin is 3 mm in thickness.

26. A rectangular poster that is 2 feet by 3 feet has a border of red that is 1/2 inch wide. Estimate the area of the border using the differential. Find the exact area of the border.

Justify the differential calculus formulae in problems 27 - 33.

Assume that u and v are differentiable functions of x.

27. d(*a* u) = *a* du where *a* is any real number.

28. d(u + v) = du + dv.

29. d(1/v) = -dv/v2.

30. d(u/v) =[vdu - udv]/v2 .

31. d(sin u) = cos u du

32. d(sec u) = sec u tan u du.

33. Suppose that w = f(u) and u = g(x) and y = f(g(x)). Prove dy = dw/du * du when interpreted appropriately.

34. Suppose g is a differentiable function with g(2)=5 and g'(2)=10. Estimate the following:
g(3) , g(1.5) , g(2.1) and g(1.99).

35. Suppose g is a differentiable function with g(1)=2 and g(2)=4. If g'(1)=4 and g'(2)=2, give two estimates for g(1.5) using the differential. Discuss briefly how these estimates relate to the true value of g(1.5)?

36. Project: Suppose L(x) is a differentiable function with L(0)=1 and for every x > -1, L'(x)=1/(1+x).

a) Estimate L(1/4).

b) Based on your estimate for $L(1/4)$, estimate $L(1/2)$.

c) Continue. Use the estimate of $L(1/2)$ and then $L(3/4)$ to estimate $L(1)$.

d) Based on this work, suggest a method to estimate $L(1)$ more accurately. Explain with an example using your method.

37. Project: Suppose $P(x)$ is a differentiable function with $P(0)=1$ and for every x, $P'(x)=3P(x)$.

a) Estimate $P(1/4)$.

b) Based on your estimate for $P(1/4)$, estimate $P(1/2)$.

c) Continue. Use the estimate of $P(1/2)$ and then $P(3/4)$ to estimate $P(1)$.

d) Based on this work, suggest a method to estimate $P(1)$ more accurately. Explain with an example using your method.