Chapter II.C Derivatives for Implicit Functions

The first thing you may be asking after reading the title of this section is "What's an implicit function? " This is not unexpected since generally courses that study functions prior to a course in calculus study only **explicit** functions. **An explicit function is a function where the value of the function for an argument is described in a specific fashion**. Usually these functions do not require any further exploration beyond the familiar list of core functions. Here are some examples of explicit functions to clarify this further:

$$f(x) = \sin(x^2), \qquad g(x) = \begin{cases} \sin(x) & when \quad x < 0 \\ \cos(x) & when \quad x \geq 0 \end{cases}$$

These equations define the functions f and g explicitly for the argument x.

$P(t) = 2^t$ and $Q(u) = 5/u$ define the functions P and Q explicitly for the arguments t and u.

One of the earliest applications of the calculus was to the study of the tangent problem for planar curves. These curves are sometimes described as **graphs of equations which are not the graphs of (explicit) functions.** Graphs of this sort sometimes are connected to models where variable quantities are changing with time and are related by some complicated equation. Even when two variables are not related as functions, it is sometimes possible to define a function that is consistent with some of the algebraic, graphical, and numerical information determined between the two variables.

Sometimes it is possible to find algebraic expressions for explicit functions (implicitly) defined in these contexts, but in many cases it is very inconvenient or even impossible to use explicit functions to solve the tangent problem with calculus. In some cases we may know the rate at which one variable is changing and wish to determine the rate at which related variables are changing. The technique we will now examine is designed to assist in these situations. It has wide application beyond the tangent problem to many contexts where variables are related by equations and information about the rates at which these variables change is the primary concern.

In the early days of analytic geometry the function notation which is common today was not used. Instead variables such as X and Y were used to keep track of quantities that were measured and equations were associated to curves through the use of coordinates. A line or familiar curve in the plane might have an equation describing it, but without any function being defined. For example, a circle with radius 5 and center at the origin (0,0) corresponds to the equation $X^2 + Y^2 = 25$, while the equations $Y^2 = X$ and $Y^2 - X^2 = 16$ describe a parabola and hyperbola. See Figures 1-3.

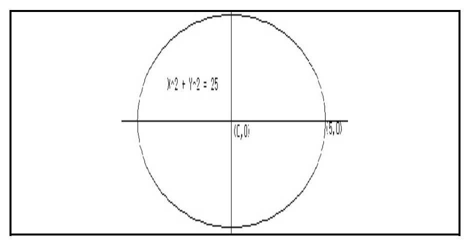

Figure 1
$X^2 + Y^2 = 25$

In the sciences as well, variables are used to describe relations between quantities in equations without any function notation. For example, the relations between **pressure P, temperature T, and volume V** of a gas are sometimes expressed by the equations $P\,V = k_1$ or $P/T = k_2$ where k_1 and k_2 are physical constants determined by the type gas and the amount of gas present and assuming either temperature or volume are held fixed. In these physical contexts it is not clear which variable is controlling the value of the other variable. Much depends on the actual context to determine whether the value of P controls V and/or T or the values of V and/or T control P.

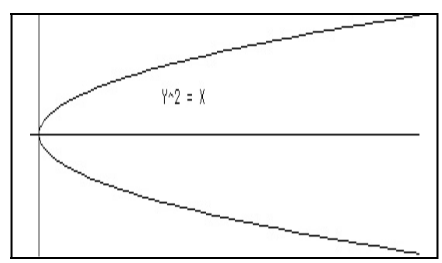

Figure 2
$Y^2 = X$

Even in geometric examples it is not clear whether Y is a function of X or X is a function of Y.

Only by conventions have we come to believe and /or accept that the second variable in an equation or on a graph is a function of the value of the first variable. **So we often suppose that Y is a function of X, name the function *f*, and write Y = *f*(X).** This does not work when for a given value of X more than one value for Y will make the equation true. **A function can have only one value for each number in its domain.** None of the previous geometric examples has an equation that makes Y an explicit function of X. The graphs of these examples also clarify this matter since they all fail to pass the "vertical line test" for graphs of functions.

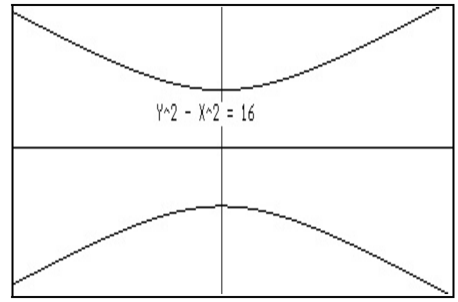

Figure 3
$Y^2 - X^2 = 16$

For example, consider the circle, $X^2 + Y^2 = 25$. When X = 3 , Y can be either 4 or -4 to satisfy the equation. For the parabola $Y^2 = X$, when X = 4, Y can be either 2 or -2, and for the hyperbola $Y^2 - X^2 = 16$ when X is 3, Y can be either 5 or -5. An even more complicated example is the equation sin(Y) = X. When X is 0, Y can be any integer multiple of π.and the equation will be true. See Figure ***.

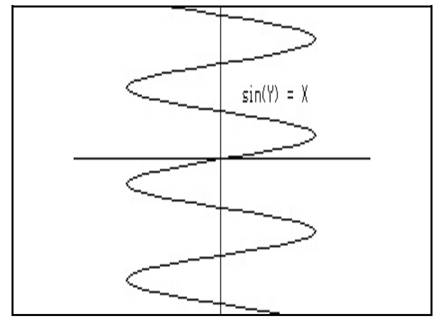

Figure 4
sin(Y)=X

Still it is not uncommon to ask for an expression of Y as a function of X that would make an equation true when used to replace Y. For the circle there are two such functions that are most apparent, namely $Y = f(x) = +\sqrt{25 - x^2}$ and $Y = g(x) = -\sqrt{25 - x^2}$. See Figure ***. But there are many more. For example,

$$Y = h(x) = \begin{cases} +\sqrt{25 - x^2} & when \quad x < 0 \\ -\sqrt{25 - x^2} & when \quad x \geq 0 \end{cases}$$

Or even more exotic:

$$Y = P(x) = \begin{cases} +\sqrt{25 - x^2} & when \quad x \text{ is a rational number} \\ -\sqrt{25 - x^2} & when \quad x \text{ is an irrational number} \end{cases}$$

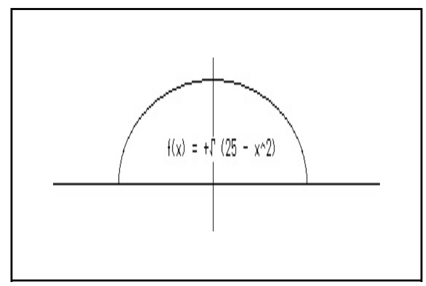

Figure 5
$Y = f(x) = +\sqrt{25 - x^2}$

Definition: We say that a function ***f* is an implicit function for Y in an equation involving X and Y** if the equation will be true whenever *f*(X) is used for the value of Y in the equation.

You can begin to see the potential for confusion in the multitude of functions that can satisfy a given equation, but this situation does allow us to clarify what we mean by an implicit function.

Remark: When considering the geometry of the graph of an equation it is **most common to consider only those implicit functions for the equation that are continuous**. The graphs of these functions will correspond to ways to dissect the equation's graph into connected pieces each of which would be the graph of one of the continuous implicit functions. This explains why in the example of $X^2 + Y^2 = 25$ the two functions *f* and *g* were easier to describe and graphically relate to the equation.

Example II.C.1: Find explicitly, two continuous functions, *f* and *g*, that are implicit functions for Y in the equation $Y^2 + 2XY + X^2 = 4$.

Solution: Notice the expression on the left side of the equation can be factored, so the equation is equivalent to $(Y + X)^2 = 4$. From this it should not be difficult to see that the following explicit functions will solve the problem: $\boldsymbol{f(x) = 2 - x}$ and $\boldsymbol{g(x) = -2 - x}$.

Example II.C.2: Find explicitly, two continuous functions, f and g, that are implicit functions for Y in the equation $Y^2+2XY+X^2-X = 4$.

Solution: Notice the expression on the left side of the equation can be factored, so the equation is equivalent to $(Y + X)^2 = X + 4$. From this it should not be difficult to see that the following explicit functions will solve the problem: $f(x) = -x + \sqrt{x+4}$ and $g(x) = -x - \sqrt{x+4}$.

Implicit Differentiation: It may seem obvious to say but **even if a function is expressed in different ways, its derivative is a single function.** As simple as this sounds, it is the key to making sense of the procedure for finding the derivative of a function defined implicitly.

We'll start our investigation of the method used to find the derivative of an implicit function with a familiar example.

Example II.C.3. Let's look again at the problem of finding a line tangent to a circle of radius 5 centered at the origin of a cartesian plane. Find the slope of the line tangent to the circle at the point (3,4)and then at the point (a,b).

Solution: Now it should be no surprise that we translate the tangent line problem into finding the slope of the tangent line. We note we can solve this problem with geometry as we did earlier before the introduction of the derivative. The tangent line is perpendicular to the radius, so its slope is -3/4.

For a separate approach we translate the problem into the language of analytic geometry and calculus. The circle has already been located in the cartesian plane, so we can describe the points on the circle by the equation $X^2 + Y^2 = 25$. Consider only the semicircle in the upper half of the plane and isolate that part of the curve involving the point (3,4).This is the graph of an explicit function., namely, $Y=f(x)=\sqrt{25-x^2}$. Interpret $f'(3)=\frac{dY}{dx}\big|_{x=3}$ as the slope of the tangent line and we have **the calculus description of the problem:**

Find $dy/dx|_{(x,y)=(3,4)}$ and $dy/dx|_{(x,y)=(a,b)}$ when y is a function of x defined implicitly by the equation $X^2 + Y^2 = 25$.

Using the chain rule we have $f'(x)=-\frac{x}{\sqrt{25-x^2}}$ and so $f'(3) = -3/4$ is again the solution to the problem..

It is even easier to use the geometry of the circle and the fact that the tangent to a circle is perpendicular to the radius as we did in Chapter *** to see that the slope of this tangent line is -3/4.

Now we'll examine the problem introducing the new technique of **implicit differentiation.** We see that in this problem Y can be thought of as a differentiable function f of X with $f(3) =4$. We consider two functions $\boldsymbol{L(x) = x^2 + y^2}$ **where** $\boldsymbol{y = f(x)}$ **and** $\boldsymbol{R(x) = 25}$.

The original equation for the circle means that for any x, $L(x) = R(x)$. Since these two functions are equal, their derivatives must also be equal. We have then that $\boldsymbol{L'(x) = R'(x)}$. We calculate these derivatives as well as we can, finding easily that $R'(x) = 0$ for all x. To find $L'(x)$ we use linearity and the chain rule.

$\boldsymbol{L'(x) = (x^2)' + [(f(x))^2]' = 2x + 2f(x)\cdot f'(x).}$

This may be more understandable in operator or Leibniz notation so we'll express this same derivation in those notations as well.

$$D_x L(x) = D_x(x^2) + D_x(y^2) = 2x + \mathbf{D_y(y^2)\,D_x(y)} = 2x + 2y\,\mathbf{D_x(y)}.$$

$$\frac{dL}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 2x + \frac{d}{dy}(y^2)\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$

Now since $R'(x) = L'(x)$ we have $\mathbf{0 = 2x + 2\,f(x)\,f'(x)}$. Evaluate this last equation when $x=3$ [and therefore $f(3) = 4$] and we find that $0 = 6 + 2\,(4)\,f'(3)$.

Solve for $f'(3)$ and we have that $\boldsymbol{f'(3) = -3/4}$.

Here is a summary of what we did, step by step:

1. **Assume that Y is a differentiable function** of x close to and at the point of interest, denoted $Y = f(x)$.

1. In this example we can actually solved for Y in terms of X to see from its form that this is indeed true.

2. In the equation that implicitly defines Y as a function of x, **treat the left hand side as a function of x, called L(x), and the right hand side also as a function of x, called R(x).**

2. $\mathbf{L}(X) = X^2 + Y^2 = X^2 + (f(X))^2$;
$\mathbf{R}(X) = 25$.

3. **Find the derivatives** of **L** and **R**.

3. To find the derivative of L we need to remember to use the chain rule.
$\mathbf{L}'(X) = 2X + 2Y\, dY/dX$
$= 2X + 2(f(X))\,f'(X)$.
Since **R** is a constant function its derivative is easy, $\mathbf{R}'(X)=0$.

4. Since the equation states, $\mathbf{L}(x)=\mathbf{R}(x)$, it must be that $\mathbf{L'(x)=R'(x)}$. This allows us to equate the results of step 3. This equation between derivatives will give a relation between Y, X, and dY/dX. **Now solve to find the value of dY/dX.**

4. Equating the results of step 3 gives
$2X + 2Y\, dY/dX = 2X + 2(f(X))\,f'(X)=0$.

Using X=3 and Y=4 we obtain the equation $2(3) + 2(4)\, dY/dX|_{(X,Y)=(3,4)} = 0$.
Solving for the derivative gives $dY/dX|_{(X,Y)=(3,4)} = -6/8 = -3/4$.

Using X=a and Y=b we obtain the equation

$$2(a) + 2(b)\, dY/dX|_{(X,Y)=(a,b)} = 0 .$$

Solving for the derivative gives $\mathbf{dY/dX|_{(X,Y)=(\boldsymbol{a},\boldsymbol{b})} = -2\boldsymbol{a}/2\boldsymbol{b} = -\,\boldsymbol{a}/\boldsymbol{b}}$.

Example II.C.4: Let's consider another familiar example from the point of view of implicit differentiation. Our problem is to find the derivative of Y where Y is an implicit function defined by the equation $Y^2 = X$ at the point where X= 4 and Y=2. We recognize that $Y = X^{1/2}$ for the values being considered, and thus we can find the result directly from our knowledge of the calculus: $dY/dX|_{X=4} = \frac{1}{2} X^{-1/2}|_{X=4} = 1/4$. We'll follow the same outline as the previous example for finding the derivative here with implicit differentiation.

1. Assume that Y is a differentiable function of x close to and at the point of interest, denoted $Y = f(x)$.

1. Again- we have already seen this is true without any assumption

2. In the equation that implicitly defines Y as a function of x, treat the left hand side as a function of x, called $L(x)$, and the right hand side also as a function of x, called $R(x)$.

2. $L(X)=Y^2 = (f(X))^2$ and $R(X)=X$.

3. Find the derivatives of L and R.

3. To find the derivative of L we need to remember to use the chain rule.
$L'(X)=2Y\ dY/dX = 2f(X)f'(X)$.

The derivative of R is easy, $R'(X)=1$.

4. Since the equation states, $L(x)=R(x)$, it must be that $L'(x)=R'(x)$. This allows us to equate the results of step 3. This equation between derivatives will give a relation between Y, x, and dY/dX from which the value of dY/dX can be found.

4. Equating the results of step 3 gives $2Y\ dY/dX = 2f(X)f'(X) = 1$.
Using X=4 and Y=2 we obtain the equation $2(2)\ dY/dX|_{(X,Y)=(4,2)} = 1$. Solving for the derivative gives $dY/dX|_{(X,Y)=(3,4)} = 1/4$.

Now let's see how implicit differentiation is described in its use.

Step 1 is often not mentioned. In this step the assumption is made that everything is okay, *i.e.*, there is a differentiable function implicitly defined by the equation being considered.This is only a little dangerous, since there is a theoretical result that says in essence that if the procedure works to give a numerical result, then the assumption in fact will be true.

Steps 2 and 3 are combined. Usually the two functions L and R are not labeled. Instead a statement is made about differentiating the equation or both sides of the equation, and then equating the derivatives of both sides of the equation.

Here is how our first example is written:

$$\frac{d}{dx}\left(x^2+y^2 = 25\right)$$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) = 0$$

$$2x+2y\frac{dy}{dx} = 0$$

and here is the second example:

$$\frac{d}{dx}\left(y^2 = x\right)$$
$$\frac{d}{dx}y^2 = \frac{d}{dx}(x)$$
$$2y\frac{dy}{dx} = 1.$$

At this stage there is a choice: use the values for X and Y to reduce the equation to an equation with only the derivative or solve for the derivative symbolically first and then "plug in" the values for X and Y. Here's the first approach [which can help avoid errors in algebra]:

$$2x+2y\frac{dy}{dx}\Big|_{(x,y)=(3,4)} = 0 \qquad 2y\frac{dy}{dx}\Big|_{(x,y)=(4,2)} = 1$$
$$6+8\frac{dy}{dx}\Big|_{(x,y)=(3,4)} = 0 \qquad 2\cdot 2\frac{dy}{dx}\Big|_{(x,y)=(4,2)} = 1$$
$$\frac{dy}{dx}\Big|_{(x,y)=(3,4)} = \frac{-6}{8} = -\frac{3}{4} \qquad \frac{dy}{dx}\Big|_{(x,y)=(4,2)} = \frac{1}{4}$$

While here is the second approach:

$$2x+2y\frac{dy}{dx} = 0 \qquad 2y\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y} \qquad \frac{dy}{dx} = \frac{1}{2y}$$
$$\frac{dy}{dx}\Big|_{(x,y)=(3,4)} = -\frac{3}{4} \qquad \frac{dy}{dx}\Big|_{(x,y)=(4,2)} = \frac{1}{2\cdot 2} = \frac{1}{4}$$

Implicit Differentiation: Here is a summary and abstract of the process used in this example to find the slope of the line tangent to the graph of the equation at the point (a,b) on the graph. This process is called **implicit differentiation.**

1. Assume that Y is a differentiable function f defined implicitly by the equation with f(a)=b.

2. Treat the left and right sides of the equations as functions of x. These functions are equal for all x because of the equation and therefore **their derivatives must being equal** as well.

3. Find the derivatives of the expressions on both sides of the equation and set these derivatives equal. The derivatives involved x, y and **y'=*f* '(*x*) = dy/d*x* = D_xy.**

4. Using the equation between the derivatives of the left and right sides of the original equation, evaluate the expressions using x = a and y = f(a) = b. This gives an equation with f '(a) as the only unknown.

5. Solve for *f* '(a), the slope of the tangent line at (a,b).

Comments: 1. It is customary when using implicit differentiation to use the Leibniz notation. This is primarily because of the ease with which this notation deals with the rules without confusion and avoids the need to name functions during the process.

2. In using implicit differentiation, the variable Y is frequently involved in applications of the product and chain rules. When these situations arise it is important to realize that we are trying to find **dy/d*x*.** Don't fail to express this derivative in the application of the rules.

3. For application of the chain rule, y plays the role of the linking variable which has usually been the variable "u" in the chain rule examples.

Here are two more examples to illustrate the method.

Example II.C.5. Find the slope of the line tangent to the graph of the equation $X^2 + 4Y^2 = 4$ at the point $(1,\sqrt{3}/2)$.

Solution: Assume there is a differentiable function f with $f(1) = \sqrt{3}/2$ that is an implicit function for y in the equation. We consider the eqaution $x^2 + 4y^2 = 4$.

Now differentiate both sides of the equation with respect to x, so that

$$\frac{d}{dx}\left(x^2+4y^2 = 4\right)$$

$$\frac{d}{dx}(x^2+4y^2) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(4y^2) = 0$$

$$2x+8y\frac{dy}{dx} = 0$$

Thus when $x = 1$ and $y = \sqrt{3}/2$ we find that

$$2x+8y\frac{dy}{dx}\Big|_{(x,y)=(1,\frac{\sqrt{3}}{2})} = 0$$

$$2+8\left(\frac{\sqrt{3}}{2}\right)\frac{dy}{dx}\Big|_{(x,y)=(1,\frac{\sqrt{3}}{2})} = 0$$

$$\frac{dy}{dx}\Big|_{(x,y)=(1,\frac{\sqrt{3}}{2})} = \frac{-2}{4\sqrt{3}} = \frac{-\sqrt{3}}{6}$$

So the slope of the tangent line is $-\sqrt{3}/6$.

Example II.C. 6. Find the point on the graph of the line $Y = 5X$ with first coordinate between 0 and 6 that is closest to the point (3,3).

Solution: Although this does not appear to be a problem of implicit differentiation, we'll use it to ease the finding of the critical points for the distance function involved in this problem. The problem asks to find the point that is **closest** to (3,1) and on the line. Let S be the distance from a point on the line $(x, 2x)$ to the point (3,3), so $S^2 = (x-3)^2 + (2x - 1)^2$. Rather than express S explicitly as a function of x, we'll differentiate implicitly to find the critical points for S in the interval [0,6]. Thus

$$D_x(S^2) = D_x\ [(x-3)^2 + (2x - 1)^2]$$

So $\quad 2S \cdot D_x S = 2(x - 3) + 2(2x - 1)(2)$.

Now the critical points will be when $D_x S = 0$, so set $D_x S = 0$ in the equation and solve for x. This gives $0 = 2(x - 3) + 2(2x - 1)(2)$. Thus $10 = 6x$ and $x = 5/3$ is the only critical point for S in the interval [0,6]. The smallest value for S will also give the smallest value for S^2 so we need only check the values of S^2 at the critical point and the endpoints of the interval. The value of S^2 at 0 is 10, at 6 it is 130, while at 5/3 it is $16/9 + 25/9 = 41/9$. Thus S is smallest when $x = 5/3$ and the point on the line that is closest to (3,1) has for its coordinates (5/3,10/3).

Application to Rate Problems: In many situations it is impossible to measure directly how a certain variable is changing, while a related variable can be observed directly. Here the investigator, like a detective, may be able to make inferences from the relations between the observed and the unobserved variables leading to a conclusion about the rate at which the unobserved variable is changing. These problems suggest the image that the Greek philosopher Plato used to describe a person in a cave trying to discern what happens outside the cave by watching the shadows on the walls of the cave.

Related rate problems frequently involve rates of change for variables that are changing with time. The next two examples illustrate how the ideas of implicit differentiation help solve these problems.

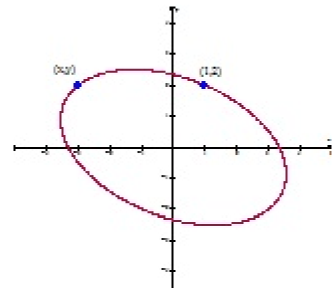

Figure 6
$X^2 + XY + 2Y^2 = 11$

Example II.C.7. Suppose an object is moving on the curve in the plane with equation $\mathbf{X^2 + XY + 2Y^2 = 11}$ so that when it is at the point (1,2) its shadow projected vertically onto the X-coordinate measured in centimeters is moving to the right at a rate of 5 centimeters per second. See Figure 6. **Find the rate at which shadow of the point projected horizontally onto the Y-coordinate is changing at the same time.**

Solution: The coordinates of the point (X,Y) are the variables that give the values of the shadows projected on the axes. Since **X and Y both depend on time, assume that each is a differentiable function of time.** The expression on the left side of the equation for the curve is a function of time which is constant because the right side of the equation equals 11. Differentiate the left side's expression as a function of time and set it equal to 0 , the derivative of the constant function 11. Using Leibniz notation we have

$$\frac{d}{dt}(X^2+XY+2Y^2)=\frac{d}{dt}(11)=0$$

So $$2\frac{dX}{dt}+X\frac{dY}{dt}+Y\frac{dX}{dt} + 4Y\frac{dY}{dt} = 0 .$$

Now we use the given information at the point (1,2), i.e., when X = 1 cm, Y = 2 cm and dX/dt=5 cm/sec in the equation to find that $2\cdot5+1\cdot\frac{dY}{dt}+2\cdot5 + 4\cdot2\frac{dY}{dt} = 0$

Thus $\frac{dY}{dt} = -\frac{20}{9} = -2\,\frac{2}{9}$ cm/sec. This should make sense at least in its sign being negative indicating that as the point is moving to the right it is also moving down on the curve at the point (1,2).

Example III.C. 8. A freight train left Albany New York heading due West for Buffalo at a speed of 30 miles per hour while a passenger train departed heading North for Montreal at 40 miles per hour. After one half hour at what speed is the distance between the two trains increasing.

Solution: Let X be the distance of the freight train from Albany while Y will be the distance of the passenger train from Albany at the same time. X and Y are both functions of t. If we use S to denote the distance between the two trains than from the Pythagorean Theorem we have that $S^2 = X^2+Y^2$. We assume that all the functions here are differentiable functions of time and differentiate both sides of the equation as functions of *t*, so that

$$\frac{d\,S^2}{dt} = 2S\frac{d\,S}{dt} = \frac{d}{dt}(X^2+Y^2)=2X\frac{d\,X}{dt}+2Y\frac{d\,Y}{dt}$$

Now use the given information to find dS/dt after one half hour. X is 15 , Y = 20 while dX/dt = 30 mi/hr and dY/dt = 40 mi/hr.

We use this information in the last equation which gives

$$2S\frac{dS}{dt} = 2(15)(30) + 2(20)(40) = 900+1600 = 2500.$$

Now an elementary use of the Pythagorean Theorem shows that S = 25 so dS/dt = 2500/50 = 50 miles per hour. Therefore the distance between the two trains is increasing at a speed 50 miles per hour.

Exercises II.C

1. For each of the following equations give two distinct functions *f* and *g* that will satisfy the equation when Y = *f*(X) and Y = *g*(X).

a. $Y^2 - 4X = 4$
b. $Y^2 + X^2 = 4$
c. $Y^2 + 4x^2 = 4$
d. $Y^2 - 4Y + X^2 = 0$
e. $Y^2 - X^2 = 0$
f. $Y^2 - X^2 = 1$

2. Draw the graph of a curve C in the plane that is not the graph of a function. Now use that graph to draw the graph of two different functions *f* and *g* which are implicit in the graph of C.

3. Use implicit differentiation to find the equation of the line tangent to the graph of each of the following equations at the indicated point.

a. $Y^2 - 4X = 1$ at (2,3)
b. $Y^2 + X^2 = 13$ at (2,3)
c. $Y^2 + 4x^2 = 25$ at (2,3)
d. $Y^2 - 4Y + X^2 = 1$ at (2,3)
e. $Y^2 - X^2 = 0$ at (2,2)
f. $Y^2 - X^2 = 5$ at (2,3)
g. $Y^3 = X^2$ at (1,1)
h. sin(Y) = X at (.5, $\pi/6$)
i. tan(Y) = X at (1, $\pi/4$)
j. $Y^2 - 3XY + X^2 = \cos(Y)$ at (-1,0)

4. For each of the following assume that dX/dt = 30 cm / sec. Find dY/dt at the indicated point.

a. $Y^2 - 4X = 1$ at (2,3)
b. $Y^2 + X^2 = 13$ at (2,3)
c. $Y^2 + 4x^2 = 25$ at (2,3)
d. $Y^2 - 4Y + X^2 = 1$ at (2,3)
e. $Y^2 - X^2 = 0$ at (2,2)
f. $Y^2 - X^2 = 5$ at (2,3)
g. $Y^3 = X^2$ at (1,1)
h. sin(Y) = X at (.5, $\pi/6$)

5. A 25 foot ladder is sliding down the wall of a building. The base of the ladder is moving 2 feet per second away from the wall when the top of the ladder is 15 feet above the ground. How fast is the top of the ladder falling toward the ground at this moment?

6. A person walks by a street light at night casting a shadow on the sidewalk. The light is mounted 10 feet above the ground and the person is 6 feet tall. If the person is walking away from the post at a rate of 5 feet per second, how fast is the person's shadow growing when the person is 8 feet from the post? How fast is the tip of the shadow's head moving away from post at this moment.7. Gasoline is pumped into a cylindrical storage tank with a circular base under constant pressure so that the volume is increasing at a rate of 8 cubic feet per minute. If the radius of the circular base is 2 feet, how fast is the height of the gasoline rising.

8. Find the coordinates of the point on the graph of $Y = X^2$ that is closest to the point (3,5). Show that the line between these two points is perpendicular to the tangent line . 9. a) Show that the graph of the equation $Y^2 + 4X^2 = 5$ has exactly two points on it with horizontal tangents and two points where implicit differentiation fails to give a value for dY/d*x*. b)Show that the graph of the equation $4Y^2 - X^2 = 3$ has exactly two points on it with horizontal tangents and no points where implicit fails to find a slope for the tangent to the graph of the equation.

c)Show that the graph of the equation $4X^2 - Y^2 = 3$ has no points on it with horizontal tangents and two points where implicit fails to find a slope for the tangent to the graph of the equation.

10. Use implicit differentiation to show that if $Y^k = X^n$ then and Y is differentiable, then dY/dX = $(n/k)X^{n/k-1}$.

11. The pressure P and the volume V of a certain amount of gas are related by the equation PV=

c where c is a constant. Suppose the volume of the gas is decreasing at a rate of 4 cubic feet per minute. When the volume is 30 cubic feet the pressure of the gas was measured as 28 lbs/square inch. How fast is the pressure of the gas changing at this moment? Is the pressure increasing or decreasing? Explain briefly.

12. Find all points on the graphs of the following equations where there is a horizontal line tangent to the curve.

a. $X^3 + Y^3 = 3XY$ [Folium of Descartes]

b. $Y^2 (1 + X) = X^2 (1 - X)$ [Strophoid]

c. $Y(1 + X^2) = 2X$ [Serpentine Curve]

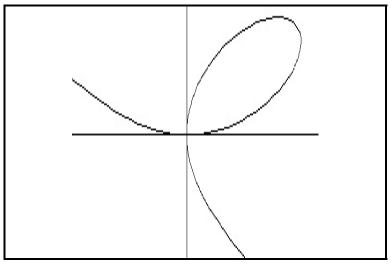

Figure 7
The Folium of Descartes

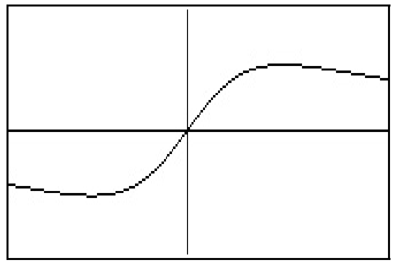

Figure 9
The Serpentine Curve

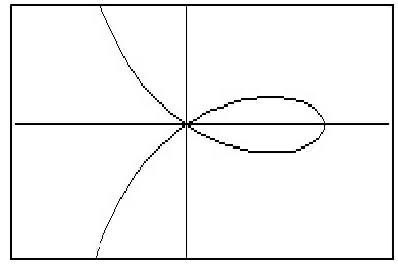

Figure 8
The Strophoid

13. A search light is rotating at a rate of two revolutions per minute. It casts a beam of light on a wall that is 100 meters away from the light. How fast is the image of the light moving on the wall at a point that is 75 meters from the point on the wall that is closest to the light. [Hint: Draw a picture and express the position of the light's beam on the wall as a function of the angle made by the beam with the line perpendicular to the wall that passes through the point where the search light is located. How fast is this angle changing?]