II. The Calculus of The Derivative

In Chapter I we learned that derivative was the mathematical concept that captured the common features of the tangent problem, instantaneous velocity of a moving object, marginal concepts in economics, and the density function of a random variable as well as rates in general. As important as the derivative is conceptually, the power of the derivative for applications comes as well from the fact that there is a method for finding the derivatives of functions expressed in fairly complicated formulae. We have seen some of this power already in the linearity properties demonstrated in I.F.

The method for systematically finding derivatives is described as **the calculus of derivatives or the differential calculus.** The object of developing this calculus for mathematicians and scientists of the 17th, 18th and 19th centuries was to make computations easier by being more mechanical and therefore requiring less thought. With rules and procedures that worked because of the form of a suitably posed problem , the calculus user can spend more time on recognizing applications and modelling, developing settings appropriate for the calculus, and taking calculus results back to applications to see that they make sense. The calculus can free the user from the drudgery of complicated and often forgotten arguments by allowing the form of a problem to drive the search for a content solution.

II.A. The Product And Quotient Rules

CAVEAT: (Warning!) It is all too easy to think that the derivative of the product is the product of the derivative. This is not true, as Example II.A.1. illustrates. The rule for finding quotients is also not simply to find the quotient of the derivatives. **Don't be trapped by thinking that everything in the calculus is simple**. The rules for products and quotients are not hard, and they make sense, but that doesn't mean they are as easy as the linearity rules.

Finding the derivatives for polynomials turned out to be fairly straight forward because of the "linearity" properties of the derivative. Unfortunately finding the derivatives for functions expressed as products and quotients is not so simple. In this section we will learn the calculus procedures for finding the derivatives of functions expressed with products and quotients. The next example illustrates that these rules are not as simple as linearity.

Example II.A.1: Find P'(0) when $P(x) = (x^2 + 5x - 3)(7x^2 + 11x + 9)$.

Solution: First we'll have to multiply the factors of $P(x)$ to express P as a polynomial in a standard form for differentiation.

$$P(x) = 7x^4 + 46x^3 + 43x^2 + 12x - 27$$

Thus $P'(x) = 28x^3 + 138x^2 + 86x + 12$ and $P'(0) = 12$.

Notice the factors of this polynomial have derivatives

$$\frac{d}{dx}(x^2+5x-3)|_{x=0} = 2x+5\ |_{x=0}=5 \text{ and}$$

$$\frac{d}{dx}(7x^2+11x+9)|_{x=0}(0) = 14x+11|_{x=0} = 11.$$

No simple formula combining 5 and 11 seems to give the result of 12.

Products of Linear Functions: In the graphical interpretation of the derivative we can consider the derivative of a function *f* at *a* as the slope of the tangent line. This line is also the linear function that best approximates f at a and its slope is the coefficient of the variable "X" in the linear function. With this point of view the following example using linear functions as factors may help you understand the product rule.

Example II.A.2: Find Q'(0) when

$$\mathbf{Q(X) = (mX + b)(nX + c) = mnX^2 + (mc + nb)X + bc.}$$

Solution: Here Q is the product of the two linear functions, $f(X) = mX + b$ and $g(X) = nX + c$. Note that $f(0) = b$, $g(0) = c$, $f'(0) = m$ and $g'(0) = n$.
Clearly, $Q'(0) = mc + nb = f'(0)\,g(0) + g'(0)\,f(0)$.

Example II.A.1 (again): Consider $P(X) = f(X) \cdot g(X)$ where

$$f(X) = X^2 + 5X - 3 \text{ and } g(X) = 7X^2 + 11X + 9.$$

The linear function best approximating f at 0 is 5X - 3 and for g at 0 it is 11X + 9. [Check this.] We look for the derivative of the product of these two linear estimators as a candidate for P'(0).

From the last Example II.A.2, the derivative of the product of the two linear estimators is given by $f'(0)g(0) + g'(0)f(0) = (5)(9) + (11)(-3) = 12$. Pretty remarkable! As was shown previously, this number is precisely P'(0). In a roundabout way we have connected P'(0) to the factors f and g.

The Product as an Area: The next example gives a dynamic interpretation recognizing the product of two functions as an area. This approach gives another way to think about the derivative of a product. It provides the key visual tool for understanding the algebra we'll use to understand the rule for finding zderivatives of products.

Example II.A.3: Consider a rectangle where the length of the sides varies with time. At time t its length is $f(t)$ and its width is $g(t)$. Thus the area of the rectangle P at time t is determined as a product: $P(t) = f(t) \cdot g(t)$. Assuming f and g are differentiable at time a, find P'(a).

Solution: To determine an estimate of the derivative of P at time a, we consider the difference quotient $\frac{P(a+h)-P(a)}{h}$.

Assuming $h > 0$, this can be thought of as the average rate of change of area over the time interval $[a, a+h]$. Now consider Figure 1 which visualizes the case where both f and g have increased during the time interval.

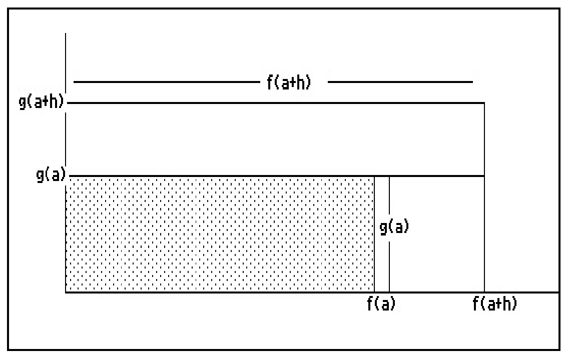

Figure 1

As usual we will organize the work of analyzing the expressions in the difference quotient using **four steps**:

Step 1: From the definition of P(t) as the area determined by $P(t) = f(t) \cdot g(t)$, we have

$$P(a+h) = f(a+h) \cdot g(a+h)$$
$$P(a) = f(a) \cdot g(a).$$

Step 2: [Subtract] $P(a+h) - P(a) = f(a+h) \cdot g(a+h) - f(a) \cdot g(a)$

In the figure, the unshaded region has area $P(a+h)$-$P(a)$. The figure shows this region composed of two distinct rectangles, leading to the algebraic equation

$$P(a+h)-P(a)=[f(a+h)-f(a)]\cdot g(a)+[g(a+h)-g(a)]\cdot f(a+h).$$

Step 3: [Divide by h.] $\dfrac{P(a+h)-P(a)}{h}=\dfrac{[f(a+h)-f(a)]}{h}\cdot g(a)+\dfrac{[g(a+h)-g(a)]}{h}\cdot f(a+h).$

This equation is true for all small h (positive or negative).

Step 4:[**<u>Think</u>**] As $h \to 0$, $\dfrac{[f(a+h)-f(a)]}{h} \to f'(a)$and $\dfrac{[g(a+h)-g(a)]}{h} \to g'(a)$(since f and g are differentiable at a). It should also make sense that $f(a+h) \to f(a)$. [This was justified in our previous discussion in Chapter I.I showing that "differentiability implies continuity." See ###] Thus, it should make sense that

$$\frac{P(a+h)-P(a)}{h} \to f'(a)\cdot g(a) + g'(a)\cdot f(a).$$

By following the **four step analysis** of the difference quotient required by the definition of the derivative, we have found that $P'(a) = f'(a)\cdot g(a) + g'(a)\cdot f(a)$.

We conclude these remarks with a statement of the **Product** Rule.

Theorem II.1:(The Product Rule.) Suppose that f and g are differentiable functions at a and that $P(X) = f(X)\cdot g(X)$ for all X. Then P is differentiable at a and $P'(a)$ is determined by the equation

$$\mathbf{P'(a) = g(a)\cdot f'(a) + f(a)\cdot g'(a).}$$

In the operator notation this is written

$$\mathbf{DP(a) = g(a)\cdot Df(a) + f(a)\cdot Dg(a).}$$

In the Leibniz notation, variable names replace function names:

y replaces P(X), u replaces f(X) and v replaces g(X). The product rule is written as

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Proof (Outline): The algebraic and limit arguments given in Example II.A.3 are valid under the hypotheses of the theorem. **EOP.**

Example II.A.4: Suppose $P(x) = (x^3 - 5x^2 + 8x - 9)(6x^4 - 2x + 6)$. Find $P'(1)$.

Solution: We let $f(x) = x^3 - 5x^2 + 8x - 9$ and $g(x) = 6x^4 - 2x + 6$ so that $P(x) = f(x)\cdot g(x)$ for all x and the product rule can be applied.

To begin applying the rule we find $f'(x) = 3x^2 - 10x + 8$ and $g'(x) = 24x^3-2$.

So we calculate easily: $f'(1)=1$, $g'(1) = 22$, $f(1) = -5$, and $g(1) = 10$. Hence by the product rule

$$\begin{aligned} P'(1) &= g(1)\cdot f'(1) + f(1)\cdot g'(1) \\ &= (10)\cdot(1) + (-5)\cdot(22) = -100. \end{aligned}$$

Example II.A.5: Suppose $P(x) = e^x (x^3 + 5x)$. Find $P'(x)$.

Solution: We let $f(x) = e^x$ and $g(x) = x^3 + 5x$ so that $P(x) = f(x)\cdot g(x)$ for all x and the product rule can be applied. We find $f'(x) = e^x$ and $g'(x) = 3x^2 + 5$. So we apply the product rule:

$$P'(x) = \quad g(x)\cdot f'(x) + f(x)\cdot g'(x)$$

$= (x^3 + 5x)e^x + e^x (3x^2 + 5)$
$= e^x (x^3 + 3x^2 + 5x + 5)$.

Example II.A.6: Suppose $P(x) = x \sin(x)$. Find $P'(x)$.

Solution: We let $f(x) = x$ and $g(x) = \sin(x)$ so that $P(x) = f(x) \cdot g(x)$ for all x and the product rule can be applied. We find $f'(x) = 1$ and $g'(x) = \cos(x)$ So we can calculate easily by the product rule

$$P'(x) = g(x) \cdot f'(x) + f(x) \cdot g'(x) = \sin(x) + x\cos(x).$$

Example II.A.7: Suppose $P(x) = \frac{x^2+3}{x}$ $(x \neq 0)$. Find $DP(x)$.

Solutions: We will solve this problem with three different approaches - each illustrating a different character of the rules.

A. [Algebra] This problem can be done without the product rule by noticing that from elementary algebra: $P(x) = x + 3/x$ so that $DP(x) = 1 - 3/x^2$.

B. [Product Rule] We apply the product rule using the fact that $P(x) = \frac{x^2+3}{x} = (x^2+3) \cdot \frac{1}{x}$.

$$\begin{aligned} DP(x) = D[(x^2 + 3)(1/x)] &= [D(x^2 + 3)](1/x) + [D(1/x)](x^2 + 3) \\ &= [2x]\,(1/x) + [-1/x^2](x^2 + 3) \\ &= 2 + (-1 - 3/x^2) \\ &= 1 - 3/x^2. \end{aligned}$$

C. [Wishful Thinking.] Let's assume that $P(x)$ has a derivative for a moment.

Certainly we can say that $x \cdot P(x) = x^2 + 3$.

We let $f(x) = x \cdot P(x) = x^2 + 3$.

Then we have two ways to find $f'(x)$ using the two different expressions.

First use $f(x) = x^2 + 3$ so $f'(x) = 2x$.

But using the product rule on $f(x) = x \cdot P(x)$ we have that $f'(x) = x \cdot P'(x) + P(x)$. Since the derivative for f doesn't depend on how we express the function, we have $2x = x \cdot P'(x) + P(x)$.

Now we solve for $P'(x)$: $2x - P(x) = xP'(x)$, so $P'(x) = 2 - P(x)/x$.

Replace $P(x)$ in the last expression with its algebraic form and we find

$$P'(x) = 2 - (x^2 + 3)/x^2 = 1 - 3/x^2.$$

Comment on Derivatives of Reciprocals:

The last [Wishful Thinking] method illustrates an interesting and important aspect of the calculus. It is one thing to show that a function is differentiable and a related but not necessarily equivalent thing to find the derivative. We will discuss this in greater depth later in this chapter. For now let's use this idea to speculate on what would be the appropriate formula for functions of the form $1/Q(x)$ by assuming these functions are differentiable.

Consider $R(x) = 1/Q(x)$.

Then 1 = R(x)·Q(x) and we use the product rule on the function $f(x)$ =1= R(x)·Q(x) to see that 0 = R'(x)·Q(x) + Q'(x)·R(x).

Solving this for R'(x) we obtain $R'(x) = -\frac{Q'(x) \cdot R(x)}{Q(x)} = -\frac{Q'(x)}{Q(x)^2}$.

Remember that we assumed that R had a derivative. We still need to prove that R has a derivative, called the reciprocal rule, using the derivative definition.

Theorem II.2: (The Reciprocal Rule) If R(x) = 1/Q(x) when Q(x) ≠ 0 and Q is differentiable at a, then R is also differentiable at a and $R'(a) = -\frac{Q'(a)}{Q(a)^2}$.

In the operator notation this is expressed as $DR(a) = -\frac{DQ(a)}{Q(a)^2}$.

In the Leibniz notation, variable names replace function names:

y replaces R(X) and v replaces Q(X), so $y = \frac{1}{v}$.

The rule is written $\frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$.

Proof: We need to consider the difference quotient $\frac{R(x)-R(a)}{x-a}$ as $x \to a$.

As usual we will organize the work of analyzing the expressions in the difference quotient using four steps:

Step 1: From the definition of R(x) we have

R(x) = 1/Q(x)

R(a) = 1/Q(a)

Step 2: [Subtract] R(x)-R(a) = 1/Q(x) - 1/Q(a)

$$= \frac{Q(a)-Q(x)}{Q(x) \cdot Q(a)}$$

$$= -\frac{Q(x)-Q(a)}{Q(x) \cdot Q(a)}$$

Step 3: [Divide by x-a.] $\frac{R(x)-R(a)}{x-a} = -\frac{Q(x)-Q(a)}{(x-a)\, Q(x) \cdot Q(a)}$.

This equation is true for all x close to but not equal to a.

Step 4:[Think] Since Q is assumed differentiable at a, as $x \to a$, and [Q(x) - Q(a)]/[x-a] → Q'(a) (since Q is differentiable at a) and it should also make sense that Q(x) → Q(a).

[Again, this was justified in our previous discussion in Chapter I.I showing that "differentiability implies continuity." See ###]

Thus, it should make sense that $\frac{R(x)-R(a)}{x-a} \rightarrow -\frac{Q'(a)}{Q(a)^2} = R'(a)$.

By following the four step analysis of the difference quotient required by the definition of the derivative, we have found that R is differentiable at a and $R'(a) = -\frac{Q'(a)}{Q(a)^2}$.

EOP

Example II.A.8. We will use the reciprocal rule to justify the derivative formula for power functions with negative integer exponents.

First , to illustrate the algebra involved, let's consider $P(x) = x^{-7} = 1/x^7$. Using the reciprocal rule with $Q(x) = x^7$ we have

$$P'(x) = -7x^6 / (x^7)^2 = -7x^{-8}.$$

In general: If $P(x) = x^n$ where n is an integer, n<0 , we let k = -n which makes k a positive integer and $P(x) = x^{-k} = 1/x^k$.

Using the reciprocal rule with $Q(x) = x^k$ we have

$$\begin{aligned} P'(x) &= -k\,x^{k-1} / (x^k)^2 \\ &= -k\,x^{k-1} / x^{2k} \\ &= -k\,x^{k-1-2k} \\ &= -k\,x^{-k-1} = n\,x^{n-1}. \end{aligned}$$

Comment: We can also find the derivatives of some other frequently encountered functions with the reciprocal rule. Among these are the secant and cosecant functions from trigonometry as well as e^{-x}. These are left as exercises for you to gain further experience with this rule.

Example II.A.8. Here is a more complicated example that puts together the product rule and the reciprocal rule to find the derivative of a quotient of polynomials.

Suppose $G(x) = \frac{x^2-x+4}{5x^2-2x+3}$. We express this quotient as a product of a polynomial and a reciprocal, so

$$G(x) = (x^2-x+4)\cdot\frac{1}{5x^2-2x+3}.$$

Let $R(x) = 1/(5x^2-2x+3)$ so that $R'(x) = -(10x-2)/(5x^2-2x+3)^2$ from the reciprocal rule. Now $G(x) = (x^2 - x + 4)\cdot R(x)$ so we can use the product rule to find that

$$\begin{aligned} G'(x) &= (2x - 1)\cdot R(x) + (x^2 - x + 4)\cdot R'(x) \\ &= (2x-1)\cdot\frac{1}{5x^2-2x+3} - \frac{(x^2-x+4)\cdot(10x-2)}{(5x^2-2x+3)^2}. \end{aligned}$$

Place this expression over the common denominator of $(5x^2-2x+3)^2$ to obtain

$$G'(x) = \frac{(2x-1)\cdot(5x^2-2x+3)-(x^2-x+4)\cdot(10x-2)}{(5x^2-2x+3)^2}$$

$$= \frac{3x^2-34x+5}{(5x^2-2x+3)^2}. \qquad \text{(II.A.\#)}$$

This last example can be generalized to give a derivative rule for any function that is expressed as a quotient of differentiable functions, i.e., a Quotient Rule. Here it is.

Theorem II.3: (The Quotient Rule) If $G(x)=\frac{P(x)}{Q(x)}$, when Q(x) ≠ 0 and P and Q are differentiable at a, then G is also differentiable at a and $G'(a)=\frac{Q(a)\cdot P'(a)-P(a)\cdot Q'(a)}{Q(a)^2}$.

In the operator notation this is expressed as $DG(a)=\frac{Q(a)\cdot DP(a)-P(a)\cdot DQ(a)}{Q(a)^2}$.

In the Leibniz notation, variable names replace function names:

y replaces G(X), u replaces P(x) and v replaces Q(X), so $y=\frac{u}{v}$.

The rule is written $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$.

Proof: Our argument here follows the idea of the previous example using the product and reciprocal rules together.

G(x) = P(x)·1/Q(x) so using the reciprocal rule to find the derivative of R(x)= 1/Q(x) and the product rule gives

$$G'(a) = R(a)\cdot P'(a) + P(a)\cdot R'(a)$$
$$= [1/Q(a)]\cdot P'(a) + P(a)\cdot[-Q'(a)/Q(a)^2]$$

So $$G'(a)=\frac{Q(a)\cdot P'(a)-P(a)\cdot Q'(a)}{Q(a)^2}.$$ EOP

Review of Example II.A.8 This example showed precisely how the general proof was derived. Here is how the quotient rule works to find the derivative of

$$G(x) = \frac{x^2-x+4}{5x^2-2x+3}.$$

We'll use the Leibniz notation.

Let $u = x^2 - x + 4$ and $v = 5x^2 - 2x + 3$. Then $\frac{du}{dx} = 2x-1$ and $\frac{dv}{dx} = 10x-2$ So applying the quotient rule we have

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$=\frac{(5x^2-2x+3)(2x-1)-(x^2-x+4)(10x-2)}{(5x^2-2x+3)^2}$$
$$=\frac{3x^2-34x+5}{(5x^2-2x+3)^2}$$

Example II.A.9. Suppose $y = \frac{x^2-1}{x^2+1}$. Find $\frac{dy}{dx}$ and $\left.\frac{dy}{dx}\right|_{x=2}$.

Solution: We use the Leibniz notation. Let $u = x^2 - 1$ and $v = x^2 + 1$. Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 2x$. So by the quotient rule we have that

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x^2+1)2x-(x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

Thus $\left.\frac{dy}{dx}\right|_{x=2} = \frac{8}{25}$.

Comments: 1. If the example had asked only for $\left.\frac{dy}{dx}\right|_{x=2}$ I would have evaluated the expressions before simplifying the algebra. [It's easy to make algebra errors; it's easier to find arithmetic errors.] Thus after finding $\frac{dy}{dx} = \frac{(x^2+1)2x-(x^2-1)2x}{(x^2+1)^2}$ I would replace x with 2 to obtain

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{(2^2+1)2\cdot 2-(2^2-1)2\cdot 2}{(2^2+1)^2} = \frac{20-12}{25} = \frac{8}{25}.$$

2. The use of Leibniz (or operator) notation makes it unnecessary to name all expressions explicitly as functions. In the last example, after identifying the numerator and denominator in the quotient we could proceed directly to write the derivative $\frac{dy}{dx} = \frac{(x^2+1)\frac{d(x^2-1)}{dx}-(x^2-1)\frac{d(x^2+1)}{dx}}{(x^2+1)^2}$.

3. In Figure *** is a graph of the function in the last example together with the line tangent at the point (2, 3/5). Our solution does seem to make sense when interpreted as the slope of this line.

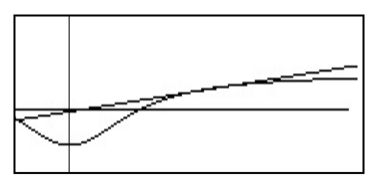

Figure 2

4. An interpretation of a reciprocal rule using mapping figures is an enlightening exercise in recognizing R(x)=1/Q(x) as the result of first finding Q(x) and then finding its reciprocal 1/Q(x). See Exercise II.A.14.

5. The derivative of the secant, tangent, cosecant and cotangent functions from trigonometry can all be found either directly from the definition of the derivative or from the reciprocal and quotient rules. See Exercises II.A.16, 17, and 18.

Exercises II.A. In problems 1 - 6 find the derivative as indicated.

1. $f(x) = (x^2 + 5)(x^3 - 6x^2 + 4x - 6)$. Find $f'(x)$ and $f'(1)$.
2. $g(t) = t(t^4 - 2t^3 + 5)(t^2 + 3)$. Find $D_t g(t)$ and $D_t g(1)$.
3. $P(u) = (u^2 - 2u)u^{-5}$. Find $P'(x)$ and $P'(1)$.
4. $Q(x) = \dfrac{x+1}{x^2+1}$. Find $Q'(x)$ and $Q'(1)$.
5. $R(x) = \dfrac{1}{x^2+2x+4}$. Find $R'(t)$ and $R'(1)$.
6. $S(t) = t^5 + 2t^3 - 4t + 8$ and $R(t)=1/S(t)$. Find $R'(t)$ and $R'(-1)$.
7. Suppose $P(x)=Q(x) \cdot G(x)$ and that Q and R are differentiable at a. Use the product rule and algebra to derive the formula for G'(a) as in the quotient rule of Theorem II.3.
8. Suppose $f(2) = 5$, $f'(2)= 7$, $g(2) = 3$ and $g'(2) = 9$.

Let $P(x)=f(x) \cdot g(x)$; $R(x) = 1/g(x)$; $S(x) = 1 / f(x)$; $V(x)=f(x)/g(x)$ and $W(x)= 1/V(x)$.
Find P'(2), R'(2) , S'(2) , V'(2) and W'(2).

9. Suppose $f(5)=2$, $f'(5) = 3$, $g'(5)=4$ and $H(x)= f(x) \cdot g(x)$ with $H'(5)= 0$. Find $g(5)$.
10. Use the product rule to find formulae for the derivatives of the following functions assuming that the functions f,g and h are all differentiable.
    a. $p_2(x)=(f(x))^2$.
    b. $p_3(x)=(f(x))^3$. [Hint: $p_3(x)=p_2(x) \cdot f(x)$.]
    c. $p_4(x)=(f(x))^4$.
    d. $p_k(x)=(f(x))^k$, k an integer , k > 1.
    e. $Q(x) = \dfrac{f(x) \cdot g(x)}{h(x)}$.
    f. $S(x)= f(x) \cdot g(x) \cdot h(x)$.
11. Find all x for which $f'(x) = 0$ when
    a. $f(x) = (x^2 - 6x + 9) \cdot (2x + 1)$.
    b. $f(x) = (x^2 - 3x) \cdot (x^2 - 9)$
    c. $f(x) = 1/(x^2 + x + 1)$
    d. $f(x) = (x - 2)/(x^2 + 1)$
12. Suppose that $P(x) = f(x) \cdot g(x)$ and $f(a) = 0$ while $f'(a) = 1$. Explain why $P'(a) = 0$ if and only if $g(a) = 0$.
13. Suppose $P(x)=(x-a)^n \cdot G(x)$ and G is function that is differentiable at a but $G(a) \neq 0$.

Prove $P'(a) = 0$ if and only if $n > 1$.

14. Draw consecutive transformation figures for $Q(x) = 3x-2$ and $G(u) = 1/u$ so that the source line for G is the target line for Q. Use the velocity interpretation for $R(x) =G(Q(x))$ in the combined transformation figure to explain why $R'(a) = Q'(a) \cdot [-1/(Q(a))^2]$.
15. Use transformation figures and a velocity interpretation to explain the reciprocal rule.
16. Use the quotient rule for tan(x) = sin(x) / cos(x) and the reciprocal rule for sec(x)=1/cos(x) to show $\tan'(x) = (\sec(x))^2$ and sec'(x) = sec(x) tan(x).
17. Use the definition of the derivative to obtain the same results as in Problem 16.

Note: Following conventional notation for trigonometric functions we write $(\sec^2(x))$ for (

$\sec(x))^2$.

18. Show that $\cot'(x) = -\csc^2(x)$ and $\csc'(x) = -\csc(x)\cot(x)$. [See 16.]

19. Find dy/dx when y is as follows:
    a. $y = x\sin(x)$
    b. $y = \sin(x)/x, \quad x \neq 0$
    c. $y = x\tan(x)$
    d. $y = x^2\sec(x)$
    e. $y=\dfrac{\sin(x)}{x^2+1}$
    f. $y = \sec^2(x)$
    g. $y = \sec(x)\tan(x)$
    h. $y=\sqrt{x}\cos(x), \quad x > 0$.
20. Find equations for the lines tangent to the graph of the tangent function at the point (0,0) and at the point ($\pi/4$, 1).
21. Find all points on the graph of the secant function between 0 and 2π where the tangent line to the graph is horizontal.
22. Use the double angle formulae, $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = \cos^2(x) - \sin^2(x)$ to show that $D_x\sin(2x) = 2\cos(2x)$ and $D_x\cos(2x) = -2\sin(2x)$.
23. Use the reciprocal rule to find the derivative of e^{-x}.
24. Use the fact that $(e^x)^2 = e^{2x}$ to show that $\frac{d}{dx}e^{2x}=2e^{2x}$. Generalize this result and justify your generalization for e^{nx} where n is a natural number.
25. Find $\frac{d}{dx}x^pe^x$ where p is any rational number.
26. Find $\frac{d}{dx}f(x)e^x$ where f is a differentiable function.
27. Find a function f where $f'(x) = x e^x$. Can you generalize? Explain how you arrived at your result.
28. The unit price $p(x)$ in dollars per pound of fish at a market is determined by the amount of fish available at the market. Assume that the demand for fish is enough that all the fish brought to market are sold.The revenues R(x) that result from the sale of x lbs. of fish at a market is the product of the weight of the fish by the unit price $p(x)$ of the fish. Use the derivative to explain why the marginal revenue will be 0 when the marginal price is equal to the opposite of the unit price divided by the amount of fish.