I. D. The Derivative of a Function [Concept Definition]

The slope of a tangent line (I.A), the instantaneous velocity (I.B), the probability density of a random variable at a point (I.C.1) and the marginal cost/revenue/profit for a change in production level (I.C.2) are four rather different applications of the same procedure for estimation. One of the most powerful qualities of mathematicians is their ability to abstract a procedure into a mathematical concept that can be applied beyond its original motivation. The development of the derivative is one important example of this kind of abstraction.

So, what is common to each of these problem situations?

- Each has a particular function f that is analyzed using estimations. The focus is on determining what happens to a quotient of differences deteremined in part by a particular number, a, in the function's domain. The function is connected to each context in a standard way:
  - the graph of an equation,
  - the position of a moving object,
  - the cumulative probability distribution of a random variable, and
  - the cost, revenue, or profit of producing a commodity.

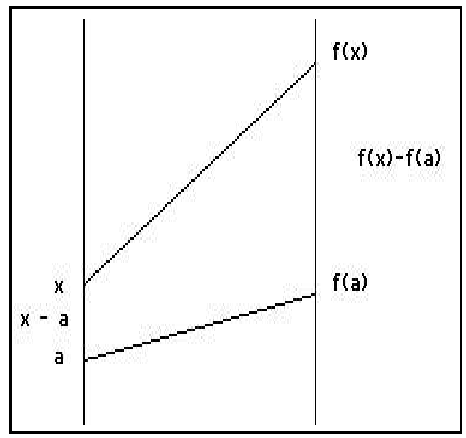

Figure I.20

- Crucial in the analysis is an estimation using the quotient of differences: the difference in the value of the function f at a second number in the domain, x, and the value f at a, $f(x)-f(a)$, divided by the difference of the two numbers, $x - a$, that is, $\frac{f(x)-f(a)}{x-a}$ where $x \neq a$. See Figure I.20. This ratio is interpreted in each context in a standard way:
  - the slope of a secant line between $(x, f(x))$ and $(a,f(a))$,
  - the average velocity of the moving object for the time interval between a and x,
  - the average probability density of the random variable being between a and x, and
  - the average marginal cost, revenue or profit for producing between a and x units of a commodity.
- If we consider $x \to a$ then this difference quotient, $\frac{f(x)-f(a)}{x-a}$, approaches a certain number L, *i.e.*, $\frac{f(x)-f(a)}{x-a} \to L$. In each context this number is interpreted in a standard way:
  - the slope of the tangent line to the graph at $(a, f(\text{a}))$,
  - the instantaneous velocity at time a,
  - the (point) probability density for the random variable at a,
  - the marginal cost, revenue, or profit for pruducing an extra unit when already producing a units of the commodity.

The number L is called the derivative of the function f at a and is denoted $f'(a)$. Let's summarize this last remark in the next box, defining this concept more explicitly.

Definition: Suppose that f is function defined on an interval that contains a.
If, when $x \to a$ and $x \neq a$, there is *one and only one number L* so that

$$\frac{f(x)-f(a)}{x-a} \to L$$

then we say that ***L* is the derivative of *f* at *a***, and write ***f* '(*a*) for *L***. We also will say in this situation that f is **differentiable at *a*.** In summary,when $x \to a$,

$$\frac{f(x)-f(a)}{x-a} \to f'(a).$$

Example I.D.1. Using the definition of the derivative, find $f'(5)$ and $f'(a)$ when

$$f(x) = x^2 - 4x + 3 .$$

Solution: To use the definition of the derivative we need to organize the expressions needed to analyze the difference quotient related to $f'(5)$, that is, $\frac{f(x)-f(5)}{x-5}$ where $x \neq 5$. We find the derivative using four steps:

Step 1: From the equation defining f, we have $f(5) = 5^2 - 20 + 3 = 8$ and $f(x) = x^2 - 4x + 3$.

Step 2: Now we find the numerator, $f(x) - f(5) = x^2 - 4x - 5 = (x - 5)(x + 1)$.

Step 3: Continuing with some algebra we proceed to simplify the difference quotient:

$$\frac{f(x)-f(5)}{x-5} = \frac{(x-5)(x+1)}{x-5} = x + 1.$$

Step 4: Finally in a fourth step we **<u>think</u>**- as $x \to 5$, $\frac{f(x)-f(5)}{x-5} = x + 1 \to 6$.

Therefore, by following this **four step analysis** of the difference quotient required by the definition we have $f'(5) = 6$.

We now proceed in complete analogy to find the derivative of f, $f'(a)$, for any particular number a. We examine the difference quotient related to $f'(a)$, that is, $\frac{f(x)-f(a)}{x-a}$ where $x \neq a$.

This time we'll organize the four steps of our work to make some of the needed calculations and algebra more clear.

Step 1: First we express $f(x)$ and $f(a)$ algebraically and align the terms to make the subtraction more recognizable.

		$f(x)$	$= x^2 - 4x + 3$ and
	-	$f(a)$	$= a^2 - 4a + 3$
Step 2:	$f(x) - f(a)$		$= (x^2 - a^2) - 4(x - a)$ Find the difference.
			$= (x - a)(x + a - 4)$. Factor the difference.

Step 3: $\dfrac{f(x)-f(a)}{x-a} = \dfrac{(x-a)(x+a-4)}{x-a} = x + a - 4$. Find the difference quotient and then simplify the algebraic expression:

Step 4: **Think!** We complete our analysis for finding $f\,'(a)$ by considering what happens to the difference quotient with $x \to a$.

Here it should make sense that as $x \to a$, $\dfrac{f(x)-f(a)}{x-a} = x + a - 4 \to 2a - 4$.

Therefore, by following these four steps for analysis of the difference quotient required by the definition we have $\boldsymbol{f\,'(a) = 2a - 4}$**.**

Notice that when $a = 5$, this result is $f\,'(5) = 2{\cdot}5 - 4 = 6$, precisely the same value we found in the first part of this example.

The Derivative and its Interpretations

In the **tangent interpretation** for the function f, we are considering the graph of the function, or the equation $Y=f(x)$.

The derivative $f\,'(a)$ is interpreted then as **the slope of the line tangent to the graph** of the function at the point $(a,f(a))$.

In the **motion interpretation** for the function f, we are considering an object moving on a coordinate line with its position at time t determined to be $f(t)$.

The derivative $f\,'(a)$ is interpreted as **the instantaneous velocity of the object** at time a.

In the **random variable interpretation** for the function f, we are considering a random variable X with f its cumulative distribution function so that $f(A)$ is the probability that X is less than or equal to A.

When this is possible, the derivative $f\,'(a)$ is interpreted as **the point probability density function for the random variable X** at a.

Exercise: Can you give an economic interpretation for the derivative in terms of marginal cost, revenue, and profit?

Comments: 1. There are four steps to these calculations based primarily analysis of the difference quotient $\dfrac{f(x)-f(a)}{x-a}$ using the definition of f.

- Step 1: Evaluate $f(x)$ and $f(a)$.
- Step 2: Find (and simplify when possible) the difference $f(x) - f(a)$.
- Step 3: Divide by $x - a$ and then simplify algebraically (when possible)

to eliminate the $x - a$ from the denominator.

- Step 4: **Think!** Finally, analyze the simplified expression to see what happens when $x \to a$, remembering that $x \neq a$. If the last expression approaches a single number then that number is $f\,'(a)$.

2. The first calculus that we encounter is a calculus for derivatives. **The purpose of this calculus is to give a systematic approach for finding the derivative of a function at a point that *does not* require the use of the definition of the derivative to compute its value.** This calculus relies on two features:

(i) knowledge of the derivative for a core list of functions and

(ii) rules for using these core functions to determine the derivative of more complicated functions by recognizing how they are constructed algebraically from core functions.

The key core functions we will explore initially in Sections I.F.1 and I.F.2 are powers of x, polynomials, the sine and cosine functions, and exponential functions.

3. It would be nice if every function had a derivative whenever it was defined, but unfortunately this is not the case as the next example demonstrates.

Symbols and Numbers. As we discussed in Chapter 0.B.1 sometimes we have a symbol that appears to represent a number when in fact there is no number that satisfies that conditions that the symbol prescribes. This possibility is often overlooked when ever use of a symbol does represent a number. We consider an example of a function that does not have a derivative to make you aware of this possibility for the number defined by the derivative. Sometimes there is no number that will satisfy the requirements to be the derivative of $f(x)$ at a, even though we write $f'(a)$.

EXAMPLE I.D.2. Suppose $\boldsymbol{f(x) = *x - 2* + 3}$. This function is defined for all real numbers. We'll focus attention on 2 and show that there is no derivative for f at 2, that is, $f'(2)$ does not exist!

Outline of the argument: To show that there is no derivative, we'll use the definition and analyze the difference quotient as we have in the other examples in this section. One special feature of this function is its use of the absolute value. Since the absolute value of a number is characterized by two cases depending on whether the number is positive or negative, our analysis of the difference quotient will also use two cases, when $x>2$ and when $x<2$. In the first case our analysis will find the difference quotient is always 1, while in the second case the difference quotient is always -1. When we think of x as a number close to 2, the difference quotient does not approach a single number. But the definition requires that the difference quotient approach one and only one number. Therefore this function cannot have a derivative.

[You might want to sketch a transformation figure of $f(x)$ for [0,4] to better understand the next analysis.]

And now for the details of the argument:

Case 1. First we consider the case when $x > 2$ and using the four steps analyze the difference quotient $\frac{f(x)-f(2)}{x-2}$. In this case $*x - 2* = x - 2$.

Step 1: $\quad f(x) = *x - 2* + 3 = x - 2 + 3 = x + 1$ and

$\quad - \quad f(2) = 3$

Step 2: $f(x) - f(2) = x - 2$

Step 3: $\frac{f(x)-f(2)}{x-2} = 1.$

If $f'(2)$ exists, this last equation suggests the only possible value for $f'(2)$ is 1.

Case 2. Now consider an x where $x<2$. In this case $*x-2* = 2 - x$.

Step 1: $f(x)$ = *x - 2*+ 3 = 2 - x + 3 = 5 - x and
- $f(2)$ = 3

Step 2: $f(x) - f(2) = 2 - x$

Step 3: . $\frac{f(x)-f(2)}{x-2} = \frac{2-x}{x-2} = -1$.

Thus no matter how close x is to 2, **if $x < 2$ the quotient that is analyzed to estimate $f'(2)$ is -1, which is not approaching 1 !**

Step 4: **Think!** To have a derivative at 2, the difference quotient $\frac{f(x)-f(2)}{x-2}$ should approach a single number for x close to 2. Since there is not a single number which the expressions $\frac{f(x)-f(2)}{x-2}$ approach as $x \to 2$, f does not have a derivative at 2.

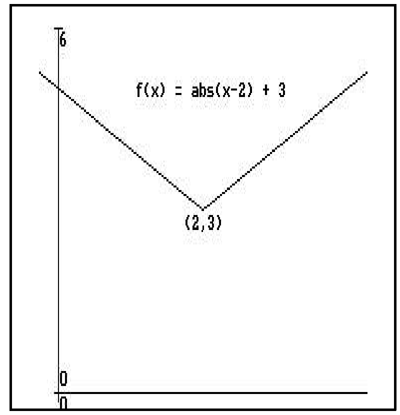

Figure I.21

Comments: 1. Figure I.21 gives the graph of $f(x) = |x-2| + 3$. Can you give a geometric interpretation for the statement that $f'(2)$ doesn't exist? What are the slopes of the secant lines? Is there a single tangent line at the point (2,3) on the graph?

2. Can you give a motion interpretation for the statement that $f'(2)$ doesn't exist? What are the average velocities? Is there an instantaneous velocity at time 2?

3. Though there are many, many functions that do not have derivatives at certain points, we will defer our discussion of these functions till later. For now we'll work on examples and contexts where there is a derivative at every point in the domain of f.

4. The notation we have used suggests that we can **think of the derivative not only as a number but also as a function, denoted f'**.

This function is called **the derivative function for f**, since f' is derived from f.

When you read "f' " try to use the phrase "the derivative of f" instead of the name of the symbols "f prime ." Though the name doesn't convey much of the meaning of the concept, it still is preferred to use the name of the concept.

The original function **f is also described sometimes as a primitive function for its derivative function f'**.

Thus for example the function $f(x)= x^2$ is a primitive function for its derivative function f' where $f'(x)= 2x$. This follows the use of these words in common language where we speak of primitive and derivative forms of art or language.

5. It is customary to describe a function by giving its value at x. Since our definition has used the variable x to find $f'(a)$ this can get a little confusing. I suggest that initially you find $f'(a)$ and then when you have understood this as a function depending on a, you can express the derivative as a function of x.

For example, when $f(X) = X^2$, you should be able to show easily that $f'(a) = 2a$. Thus once you've understood the content of the last equation it should be easy to find $f'(-4) = -8$, $f'(3)=6$, $f'(s) = 2s$ and $f'(x) = 2x$. One can continue this evaluation with any symbols you like. So

$f\,'$(anvil) = 2 anvil and $f\,'$(good)= 2 good.

Exercises I.D. For the functions in problems 1 through 7, use the definition to find the derivative at the indicated point(s).

1. $f(X) = 5X^2 - 2X$; a. 1 b. 3 c. a.
2. $f(X) = X^2 - X + 1$; a. 1 b. -1 c. s.
3. $f(X) = X^3 - 2X^2$; a. 1 b. -2 c. *t*.
4. $f(X) = X^3 - X^2$; a. 1 b. 2 c. *t*.
5. $f(X) = 1/X \;\; X \neq 0$; a. 1 b. 2 c. *a*.
6. $f(X) = 1/(X^2) \;\; X\neq 0$; a. 2 b. *x*.
7. $f(X)=\sqrt{X+1} \;\; X > -1$; a. 8 b. *a*.

8. Suppose $f(X)=\sin(X)$. Using your calculator estimate the derivative at a = 0, π, and 1. Discuss briefly what you think the derivative of the sine function is based on the tangent interpretation and a sketch of the sine function's graph.
9. For each of the following functions use your calculator to estimate the derivative at 0, 1, and 2.
   a. $f(X) = 2^X$.
   b. $f(X) = 3^X$
   c. $f(X) = (.5)^X$
   d. $f(X) = (1/3)^X$.

10. Find any points *t* where $f\,'(t) = 0$ for the functions in problems 1 - 3.
11. Find any points *t* where $f\,'(t) > 0$ for the functions in problems 1 - 3.
12. Using the definition find $f\,'(x)$ for each of the following examples :
    a. $f(x) = 3x + 2$;
    b. $f(x) = x^2 + 3x + 2$;
    c. $f(x) = 1/x + 3x + 2$.

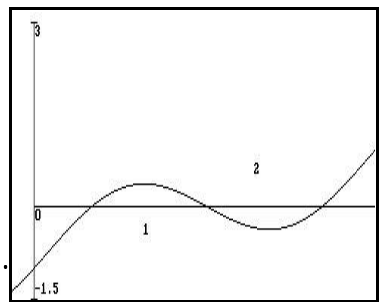

Figure I.22

13. The graph of a function f is given in Figure I.22, estimate $f\,'(1)$ and $f\,'(2)$. Based on the graph of f sketch a graph of the function $f\,'$.

14. In Figure I.23 are three graphs of functions where $f\,'(x)=g(x)$ and $g'(x)=p(x)$. Which graph is the graph of f, of g, and of p.

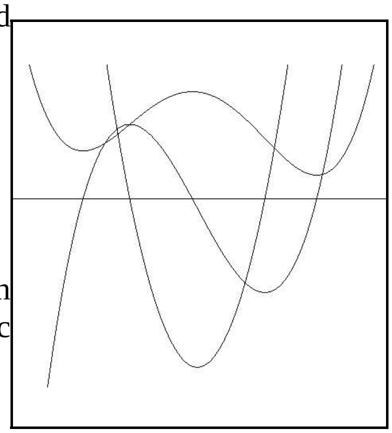

Figure I.23

15. Suppose $f(x) = x^2 + 3x$ for $0< x <2$.
    a. Find $f'(c)$.
    b. Find a numerical value for c so that $f'(c) = \dfrac{f(2)-f(0)}{2}$.
    c. Interpret this equation and the information it contains first with the geometric model of the tangency and then with the dynamic model of motion on a straight line.

For the following functions assume f has a derivative at a.

16. Suppose $g(x)=f(x)+C$ where C is some constant. Using the definition show that g also has a derivative at a and that $g'(a)=f'(a)$. Write a brief explanation of this result based on the tangency interpretation first and then on the motion interpretation.
17. Suppose $s(x) = \alpha f(x)$ where α is a constant. Show that s also has a derivative and that $s'(a) = \alpha f(a)$. Write a brief explanation of this result based on the tangency interpretation first and then on the motion interpretation.
18. Suppose $m(x) = f(x + 3)$. Show that $m'(a-3) = f'(a)$.

19. Suppose $f\,'(a) = 0$. Write a brief explanation of this result based on the tangency interpretation first and then on the motion interpretation.

20. Suppose $f(x)$ = c, a constant. Show $f\,'(x) = 0$ for all x.Write a brief explanation of this result based on the tangency interpretation first and then on the motion interpretation.

21. Suppose $f(x) = mx + b$. Show $f\,'(x) = m$ for all x.Write a brief explanation of this result based on the tangency interpretation first and then on the motion interpretation.

22. Suppose $f(x) = A x^2 + B x + C$. Show that $f\,'(x) = 0$ if and only if $x = -B/2A$. Interpret this geometrically. [Where is the vertex of the parabola that is the graph of f? Why?]