I.I Derivatives and Continuity

In this section we will explore an important consequence for a function having a derivative. We will see that this property, **called continuity**, is a **necessary** condition for having a derivative, but is **not sufficient** to ensure the existence of a derivative. Continuity is something we often take for granted in the way we think about many phenomena.

When we consider how variables change we usually think that these changes in values pass through all the possible values in between the start and the finish. As with a runner in a race or a car on a trip, the change we envision develops from the accumulation of smaller changes. these are typical of continuous variables.

The philosophical issues of the nature of physical motion have been discussed as far back as Zeno's paradoxes. These cynical paradoxes aimed at attacking logical argument as a way to discover the truth purport to demonstrate the impossibility of motion!

Variables that do not change continuously have sudden displacements like those sometimes used in physics to explain small particle motion or in business and economics where changes must come in steps because of the indivisibility of variables units like a single car or the price of a share on the stock market.

In the graphical interpretation of a function's continuity appears in the visual illusion created by the physical graphs drawn with pencils or with technology. Looking at the graph we see an image of a the curve that has no holes in it, apparently connected in one piece even though when examined more closely (perhaps with a microscope) this is not the case. A discontinuity in the function would appear in the graph as a hole or a break in the curve requiring someone tracing the curve with a pen to lift the pen off the paper to make an accurate sketch.

The Greek philosophers also considered what the nature of a continuum was. Aristotle said that a quantity was either discrete or continuous and then proceeded to try to distinguish between these two concepts. Many mathematicians and philosophers of science continue to try to explain the nature of the continuous to this day.

The word "continuity" itself suggests continuation, the connection of actions or processes developing over time. As we develop a technical meaning for continuity and being continuous, keep in mind that in doing so we are trying to capture some of the common meanings of the word as well. We would like the common usage and the technical meanings to be consistent in allowing us to describe experiences we believe are continuous as also being continuous in the technical sense. But be aware that **technical definitions of terms can lead to some strange examples that may not seem as meaningful.** In this section we will focus on the more sensible aspects of continuity and leave more subtle examples for another time.

You might want to review Example I.D.2 now, as this is our only textual example so far of a function that **we might describe naively as continuous but which does not** have a derivative at a point. See Figure ***.

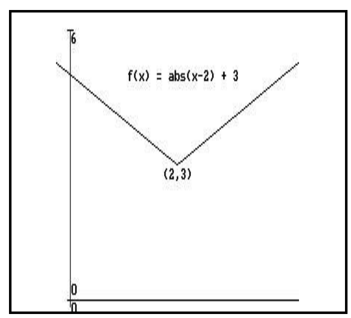

Figure 1

In section I.I.1 we'll explore the concept of continuity in its relation to a function having a derivative, in its interpretation with graphical and dynamic models, and in its precise definition. Then in I.I.2 and I.I.3 we will look more carefully at two important consequences of continuity for a function restricted to a closed interval: the intermediate value property and the extreme value property. Though reading some of this discussion may be postponed till Chapter III, you should read Section I.I.1 now, even if only quickly, since it provides a conceptual background for understanding some of the arguments presented in Chapter II.

I.I.1. Continuity and the Derivative: An Introduction. We'll begin informally by considering a question of estimation. When we find the derivative of a function f at $x = a$, we have assumed that a is in the domain of the function. How else could we make sense of the difference quotient $\frac{f(x)-f(a)}{x-a}$? A less obvious but also necessary feature of the derivative's existence is that $f(x)$ should make sense for all x sufficiently close to a, that is, for all x in some open interval containing a.

The issue of continuity arises by asking **how close $f(x)$ is to $f(a)$ when x is close to a.** We will examine interpretations of this issue later, but first let's settle the question **assuming that f has a derivative at $x = a$.** Under this assumption we know that when $x \to a$,

$$\frac{f(x)-f(a)}{x-a} \to f'(a).$$

Now in terms of estimation this means that when $x \approx a$, $\frac{f(x)-f(a)}{x-a} \approx f'(a)$. Continuing to think in terms of estimation, we have the product $\frac{f(x)-f(a)}{x-a} \cdot (x-a) \approx f'(a) \cdot (x-a)$

OR $$f(x) \approx f(a) + f'(a)(x-a).$$

But when $x \to a$, $x-a \to 0$ so $f(x) \approx f(a)$ when $x \approx a$. We summarize this work in the following very important result.

Theorem I.I.1: If f is defined in an open interval containing a, and f **has a derivative at a, then** $\lim_{x \to a} f(x) = f(a)$.

Proof: Although the preceding argument should be convincing, here is another argument that is a little more precisely organized.

One can check that for $x \neq a$, $f(x) = f(a) + \frac{f(x)-f(a)}{x-a} \cdot (x-a)$.

As $x \to a$, $\frac{f(x)-f(a)}{x-a} \to f'(a)$ and $x - a \to 0$ so the product $\frac{f(x)-f(a)}{x-a} \cdot (x-a) \to 0$.

Hence $f(x) \to f(a)$. **EOP.**

Alternative statements for the conclusion of this theorem are that

$$\lim_{h \to 0} f(a+h) = f(a) \text{ or } \lim_{\Delta x \to 0} f(x+\Delta x) = f(x).$$

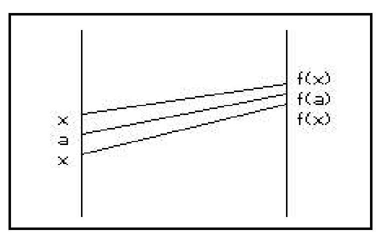

Figure 2

Comment: The conclusion of this result seems almost too natural if one considers most of the functions we've examined so far in our work with the derivative. You may have trouble imagining a function for which the conclusion does not hold. As we proceed with a discussion of the interpretation of this condition, you should begin to see that many functions do not satisfy this condition.

Interpretations: Motion. Suppose $f(t)$ denotes the coordinate of an object on a coordinate line at time t. Then the conclusion of the theorem can be interpreted as saying that when the time, x, is close to a, the position of the object, $f(x)$, will be close to $f(a)$. See Figure ***. This seems obvious when considering a physical object in motion, but not so when considering the position of a light's image which can be abruptly deflected. In futurist scenarios like "StarTrek" the technology can transport objects and people over long distances in an instant. In common language, the physical motion of an object is described as continuous. This is one reason why the term continuous is used to describe the condition found in the conclusion of the theorem.

Geometry of Curves. When we (or a graphing calculator or a computer) draw a curve we produce a physical image that represents a relation. When this curve is drawn without a break in its image, we describe this in common language as a continuous curve. When the curve is the graph of a function, then the conclusion of Theorem I.I.1 may be interpreted as saying that when x is close to a, the corresponding point on the graph $(x,f(x))$ will be close to the point $(a,f(a))$. Thus the curve will appear to be a continuum in a vicinity of the point $(a,f(a))$.See Figure 2.

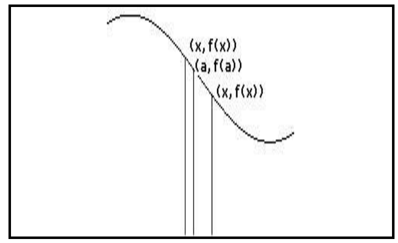

Figure 3

Here is the definition for continuity [first given in this form by the French mathematician Augustin-Louis Cauchy (1789-1857)].

> **Definition:** Suppose f is defined on an open interval containing a.
> We say that ***f* is continuous at *a*** if
> $f(x)$ makes sense when $x = a$; $\lim_{x \to a} f(x)$ makes sense; and
>
> $$\lim_{x \to a} f(x) = f(a).$$

Comment: With this definition it is now possible to paraphrase the result of Theorem I.I.1: "If f is differentiable at a then f is continuous at a," or in a single clause we can say

"Differentiability implies continuity at a point."

To understand the concepts involved in continuity better let's look at some examples where a function is **discontinuous (**not continuous) at a point. This feature of a function is often apparent on the graph of the function if sufficient detail is shown. Note that because of Theorem I.I.1 any example where the function is discontinuous at ***a*** will not have the function being differentiable at ***a*.** That would contradict the theorem's conclusion. We'll record this observation here as

Corollary I.I.2: If f is a function defined on an open interval containing a and f is not continuous at a, then f is not differentiable at a.

Proof: This is a logical consequence of Theorem I.I.1. **EOP**

Example I.I.1. Suppose $f(x)=\frac{x^2-4}{x-2}$ when $x \neq 2$ and $f(2) = 5$. We examine f at (x=)2 and note that the function is defined at 2 since we are told $f(2) = 5$. When $x \neq 2$, $f(x)$ can be simplified by algebra, giving $f(x) = x+2$.

Thus as $x \to 2$, with $x \neq 2$, $f(x) \to 4$.

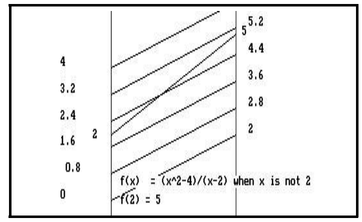

Figure 4

Unfortunately the limit is not the same as $f(2)$. So the definition of continuity is not satisfied by f at $x = 2$. See Figures *** and *** for a transformation figure and a graph to help visualize this example.

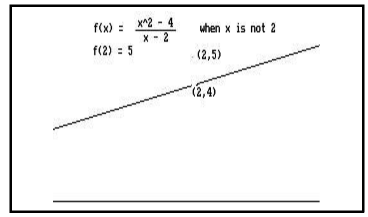

Figure 5

Comment: In the last example the discontinuity at 2 is typical of what is described as a "**removable discontinuity.**" What makes it **removable** is the fact that as $x \to 2$, $f(x)$ did have a limit, namely 4. If $f(2)$ had been 4, then $f(x)$ would have been continuous at 2. In this sense the discontinuity at 2 could be removed by considering a second function g where $g(x) = f(x)$ when $x \neq 2$ and with $g(2) = 4$. This removes the hole in the graph of f by letting $g(2)=4$. Otherwise g agrees in value with f. In fact, $g(x) = x + 2$ for all x .

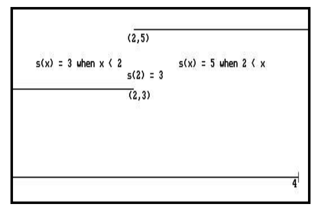

Figure 6

Example I.I.2. Suppose $s(x) = 3$ when $x \leq 2$ and $s(x) = 5$ when $2 < x$.

We consider the continuity of s at 2. Certainly $s(2)=3$, and when $x < 2$ and $x \to 2$, $s(x) = 3$. But when $x>2$ and $x \to 2$, $s(x) = 5$ which means that 3 cannot be a limit for s as $x \to 2$. But just as well 5 cannot be a limit for s as $x \to 2$. There is no limit for s as $x \to 2$. Thus the equality that is required for continuity has one expression in it that doesn't make sense, as there is no limit for $s(x)$ as $x \to 2$. This function cannot be continuous at 2. See Figures *** and *** for both the transformation figure and graph of s.

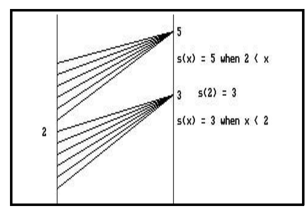

Figure 7

Remark: The function s is typical of many real situations where the value of a function will change dramatically at a particular point, sometimes called a **threshold.** Examples of this type of real situation are the prices for postage, the tax rates for personal income tax, and the assignment of grades in some (calculus) courses based on the total number of points received on tests during the course. The graph of such functions appear to make a sudden change at the threshold point. These discontinuities are described as **"step discontinuities."** What characterizes these discontinuities is the fact that as $x \to a$ with $x < a$, there is a number L where $f(x) \to L$, while as $x \to a$ with $a < x$, there is a different number R where $f(x) \to R$. Since $L \neq R$ there is no possible limit for $f(x)$ as $x \to a$. In a sense the values of the function at ***a*** take a step from L to R as x moves from less than ***a*** to greater than ***a.***

Notation for one sided limits: In the situation just described when as $x \to a$ with $x<a$, $f(x) \to L$ we say that $f(x)$ approaches L as "x goes up to a," or as "x approaches a from the left," or as "x approaches a from below."

In the limit language we say L is the **left handed limit** of $f(x)$ as $x \to a$ from below, and we write either $\lim_{x \to a^-} f(x) = L$ **or** $\lim_{x \uparrow a} f(x) = L$**.** Similar language and notation are used for the situation where as $x \to a$ with $x > a$, $f(x) \to$ R, called the right hand limit of $f(x)$ as $x \to a$ from above. That is $\lim_{x \to a^+} f(x) = R$ **or** $\lim_{x \downarrow a} f(x) = R$.

So in the last example we write $\lim_{x \to 2^-} s(x) = 3$ and $\lim_{x \to 2^+} s(x) = 5$.

Using this notation for "**one sided limits"** leads to the following

Proposition I.I.3. Suppose f is defined in an open interval containing a. The following statements are equivalent:

i) $\lim_{x \to a} f(x) = L$;

ii) $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

Discussion: This is merely expresses what should be informally clear with the notation of one sided limits. **EOP.**

It should go without saying that if a function is not defined at a it cannot be continuous at a. But the example of a removable discontinuity suggests that in some cases a function can be modified slightly by defining a new function at a so that this extended function is continuous at a. The step function example gives a situation where it is impossible to define an extension or modification of a function to be continuous at a point. Our last example in this part of section I.I shows a more severe problem with a function that "blows up" at a point.

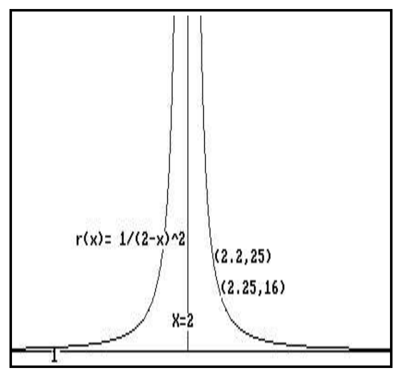

Figure 8

Example I.I.3. Suppose $r(x) = 1/(x-2)^2$ when $x \neq 2$ and r(2) = 5. When we consider the values of $r(x)$ as $x \to 2$ it is clear that these values get larger and larger. See Table ***. In fact when x is closer than .1 to the number 2, $r(x) > 100$. It shouldn't be hard to see how close you would need to be before you could say $r(x) > B$ for any number B. With this happening to the values of $r(x)$ when x is close to 2, it is impossible for $r(x)$ to have even a one sided limit as $x \to 2$. Therefore $r(x)$ is not continuous at 2 and certainly there is no way to adjust the value of r at 2 to make r continuous at 2. See Figure *** for the reason why this type of discontinuity is referred to as "blowing up." In the graph of r, the vertical line $x = 2$ is called a **vertical asymptote** for r since the graph of r appears to get closer and closer to this line as you investigate points on the graph with first coordinate close to 2.

x	$\frac{1}{(x-2)^2}$
1.5	4
1.9	100
1.99	10000
2.5	4
2.1	100
2.01	10000

Notation for Blowing Up: The fact that as $x \to 2$, r(x) gets larger and larger without any bound is denoted by writing $\mathbf{r(x)} \to \infty$. Note that ∞ in this notation is not a number, but merely indicates the tendency of r(x) to become very large without bound. With the limit notation this fact is expressed by writing $\lim_{x \to 2} r(x) = \infty$.

Variations on this use of the limit with the symbol ∞ and $-\infty$ are explored further in the exercises, as well as in later chapters.

The function *f* where $f(x) = 1/x$ when $x \neq 0$ allows us to see one sided limits together with blowing up . Draw a sketch of the graph and the transformation figure for this function to help understand why $\lim_{x \to 0^+} f(x) = \infty$ while $\lim_{x \to 0^-} f(x) = -\infty$.

Final notes on differentiability and continuity: Don't make the mistake of identifying Theorem I.I.1 with the definition of continuity. As a simple and memorable example you should be able to explain why $\boldsymbol{f(x) = *x*}$ **is continuous at 0 but not differentiable at 0**. **Continuity is a different concept from differentiability** and you will never confuse one with the other if you keep this example at hand.

Our next definitions are convenient to describe functions that are continuous at all points in a variety of situations.

Definition: We say that *f* is continuous on an open interval (*b*,*c*) if *f* is **continuous at *a* for any *a*** which is an element of (b,c).

We say that ***f* is continuous on a closed interval [*b*,*c*]** if *f* is **continuous on (*b*,*c*) and**

i) *f* is defined at *b* and *c*,

ii) when $x \to b$ with $x>b$, $f(x) \to f(b)$, that is, $\lim_{x \to b^+} f(x) = f(b)$, **and**

iii) when $x \to c$ with $x<c$, $f(x) \to f(c)$, or $\lim_{x \to c^-} f(x) = f(c)$**.**

We say that *f* is **continuous** if it is continuous for every open or closed interval in its domain.

Example I.I.4: The following functions are continuous: x^p for any rational number p, all polynomial functions, simple exponential and logarithmic functions, sin(x), and cos(x).

As we explore more functions we will note any exceptions to the continuity issue where the derivative fails to exist. Generally however we will rely on Theorem I.I.1 to explain why most of the functions we encounter in this text are continuous.

Exercises I.I.1

1. For each of the following functions draw a transformation figure and a graph illustrating the function for an interval in its domain containing the number 2. Discuss the continuity of the function at $x = 2$ using the definition.

a. $f(x) = \begin{cases} \dfrac{x^2-3x+2}{x-2} & if\ x \neq 2 \\ 5 & if\ x = 2 \end{cases}$ c. $s(x) = \begin{cases} 3x+4 & if\ x<2 \\ 10 & if\ x=2 \\ x-4 & if\ x>2 \end{cases}$

b. $f(x) = \begin{cases} \dfrac{x^3-8}{x-2} & if\ x \neq 2 \\ 3 & if\ x = 2 \end{cases}$

2. Find a value for k so that the following functions are continuous at $x = 2$. Justify your conclusion using the definition of continuity.

a. $f(x) = \begin{cases} 2x+1 & if\ x<2 \\ k & if\ x=2 \\ 3x-1 & if\ x>2 \end{cases}$ b. $p(x) = \begin{cases} kx+1 & if\ x \leq 2 \\ x^2-1 & if\ x>2 \end{cases}$

3. Suppose G(x) is defined by $G(x) = \begin{cases} x-1 & if\ x<2 \\ 7 & if\ x=2 \\ 2x-1 & if\ x>2 \end{cases}$

Find the following limits when possible:

a. $\lim_{x \to 1} G(x)$ b. $\lim_{x \to 3} G(x)$

c. $\lim_{x \to 2^-} G(x)$ d. $\lim_{x \to 2^+} G(x)$ e. $\lim_{x \to 2} G(x)$

f. Is G continuous at $x = 1$? 3? 2? Explain briefly.

4. Suppose P(x) is defined by $P(x) = \begin{cases} x^2-1 & if\ x<2 \\ 7 & if\ x=2 \\ x+2 & if\ x>2 \end{cases}$

Find the following limits when possible:

a. $\lim_{x \to 1} P(x)$ b. $\lim_{x \to 3} P(x)$

c. $\lim_{x \to 2^-} P(x)$ d. $\lim_{x \to 2^+} P(x)$ e. $\lim_{x \to 2} P(x)$

f. Is P continuous at $x = 1$? 3? 2? Explain briefly.

5. Suppose r(x) is defined by $r(x) = \begin{cases} x^3 - 1 & if\ x<2 \\ 7 & if\ x=2 \\ 5x+2 & if\ x>2 \end{cases}$

Find the following limits when possible:

a. $\lim_{x \to 1} r(x)$ b. $\lim_{x \to 3} r(x)$

c. $\lim_{x \to 2^-} r(x)$ d. $\lim_{x \to 2+} r(x)$ e. $\lim_{x \to 2} r(x)$

f. Is r continuous at $x = 1$? 3? 2? Explain briefly.

6. Find the following limits when possible.

a. $\lim_{x \to 2} (x^3 - 8)/(x - 2)$ b. $\lim_{x \to 2^+} (x^3 - 6)/(x - 2)$

c. $\lim_{x \to 2^-} (x^3 - 6)/(x - 2)$ d. $\lim_{x \to 2} (x-2) / (x^2 - 4)$

7. **Project: One sided derivatives.** If we replace the general limit with a left handed or right handed limit in the definition of the derivative of a function at a, we arrive at the notion of a one sided derivative for a function. Write a definition for the left hand derivative of f at a, denoted $f'_-(a)$. Write a definition for the right hand derivative of f at a, denoted $f'_+(a)$.

Give an example where these one sided derivatives exist for a function but where the function does not have a derivative.

Discuss the following statement with examples: A function has a derivative if and only if (i) it has a left hand derivative and a right hand derivative; and (i) the left hand derivative the derivative equlas the right hand derivative.

8. Consider the function $f(x) = \sqrt{x}$.

a. Explain why f is continuous on the interval $(0,\infty)$. Is f continous on $[0,\infty)$?

b. Using the definition of the derivative, explain why f is not differentiable at $x = 0$. Discuss this statement using the motion and the tangent line interpretations of the derivative.

9. **Project: Vertical tangent lines. Definition:** We say that the graph of a function f has a **vertical tangent line at $x = a$** if

i) f is not differentiable at $x = a$ but is differentiable for all other x in an open interval containing a, and

ii) if $\lim_{x \to a} *f'(x)* = \infty$.

a) Show that $f(x) = x^{1/3}$ has a vertical tangent line at $x = 0$.

b) Show that $f(x) = x^{2/3}$ has a vertical tangent line at $x = 0$.

c) Give an explanation of this definition in terms of the graphs of f for parts a) and b)