I.F Finding the Derivative of Some Key Functions
Draft Version 9/23/05

It is not the definition of the derivative alone that makes calculus such a widely used tool for an ever growing number of applications. The feature of the derivative that has made it so powerful and important is the ease with which it can be computed symbolically or numerically estimated. Based on some key functions and some fairly simple rules it is actually mechanical to find the derivatives of elementary functions. These elementary functions are built up from the key algebraic and transcendental functions we have seen in Chapter 0 using basic arithmetic and the ability to compose function upon function.

In this section we will lay the foundations for the symbolic calculus of derivatives by investigating the derivatives of **power functions**, some key **trigonometric functions**, namely the sine and cosine functions, and simple **exponential functions**. We will also establish two important **rules for combining functions by addition and re-scaling** function values by constant multiples. These rules will allow us to find derivatives for a large class of functions, including all polynomial functions. We will leave more complicated functions, such as the logarithmic functions, and rules for dealing with multiplication, division, and composition for later exploration in Chapter II.

I.F.1*a* Derivatives of Powers of *x*.

We've now seen several examples of how to find the derivative of a function. Two equivalent formulas for the derivative of a function f at a are the key for this work, namely,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

and $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Here is a brief summary of some results we've obtained in our previous work and exercises in this chapter.

primitive function: $f(x)=$	derivative function: $f'(a)=$
c	0
$\mathbf{m}x+\mathrm{b}$	$\mathbf{m}$
x^2	$\mathbf{2}a$
x^3	$\mathbf{3}a^2$
$\frac{1}{x} = x^{-1}$	$-\frac{1}{a^2} = (-1)a^{-2}$
$\sqrt{x} = x^{.5}$	$\frac{1}{2\sqrt{a}} = .5\, a^{-.5}$

The last four of these can be generalized to the following:

Theorem I.F.1: (**The Power or the "Down - One" Rule**)
If $f(x) = x^p$ then $f'(x) = p\,x^{p-1}$ as long as $x \neq 0$ when p -1 < 0.

This result can be written more compactly in either the Leibniz or the Operator notations as follows:

$$\frac{d(x^p)}{dx} = p \cdot x^{p-1}$$

$$D_x(x^p) = p \cdot x^{p-1}$$

We **will not prove this theorem at this stage** in all its generality. But we will prove almost as much using the techniques we have. Beginning with only positive integers for exponents **we will demonstrate the result for all power functions that have rational number exponents.**

Suggestion: If any of the following proofs becomes too abstract for you to follow, substitute a specific number for n (say n = 5) and see how the algebra works for that number. It is generally not a good idea to continue reading an argument for long without understanding the algebra being used.

You might ask, "Why bother to justify this formula? Can't we just use it and accept that it must be correct because it's written in the book?"

Proposition: I.F.2. (The Power or the "Down-One" Rule for Positive Integers.)

For any positive integer, *n*, that is, for *n* = 1, 2, 3, ...,

if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

Proof: Note first that **when $n = 1$** the result states that when $f(x) = x^1$, $f'(x) = 1x^{1-1} = x^0 = 1$. A quick look at the difference quotient here, $\frac{x-a}{x-a} = 1$, and a thought about the velocity interpretation of f that says that an object moving with its position determined by this function should help make this clear.

To help you follow this argument, let's use n=5 and write the comparable statements that will be used for the general case. The numerator of the difference quotient is

$$x^5 - a^5 = (x-a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4).$$

Thus we see that

$$\frac{f(x)-f(a)}{x-a} = \frac{x^5-a^5}{x-a} = \frac{(x-a)(x^4+x^3a+x^2a^2+xa^3+a^4)}{x-a} = x^4+x^3a+x^2a^2+xa^3+a^4$$

Now it's **time to think** about this difference quotient as $x \to a$. It has exactly 5 summands and each of these approaches a^4 as $x \to a$. So the difference quotient approaches $5a^4$, and therefore $f'(a) = 5a^4$.

When $n = 2$, the result says that when $f(x) = x^2$, then $f'(x) = 2x^{2-1} = 2x^1 = 2x$.

Also when **$n = 3$**, $f(x) = x^3$, then $f'(x) = 3x^2$. The calculations and analysis for these results were made in Chapter I.A using the tangent line interpretation of the derivative for motivation. We'll prove the general result now using the definition of the derivative from Chapter I.D,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

As in the previous section we'll employ the four step method with algebra comparable to that used for x^2 and x^3 to simplify and analyze the difference quotient $\frac{f(x) - f(a)}{x - a}$.

Step 1: For $n > 1$

$$\begin{aligned} f(x) &= x^n \text{ and} \\ - \quad f(a) &= a^n \end{aligned}$$

Step 2: $f(x) - f(a) = x^n - a^n = (x-a)\cdot(x^{n-1} + x^{n-2}\cdot a + \ldots + x\cdot a^{n-2} + a^{n-1})$.

Step 3:

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{x^n - a^n}{x - a} \\ &= \frac{(x-a)\cdot(x^{n-1} + x^{n-2}\cdot a + \ldots + x\cdot a^{n-2} + a^{n-1})}{x - a} \\ &= (x^{n-1} + x^{n-2}\cdot a + \ldots + x\cdot a^{n-2} + a^{n-1}). \end{aligned}$$

Step 4: Think! Now it's **time to think** about this difference quotient as $x \to a$. The expression for the quotient has exactly n summands and each of these approaches a^{n-1} as $x \to a$. So the difference quotient approaches na^{n-1}, and therefore $f'(a) = n\, a^{n-1}$.

In summary, we'll write the four steps of of this analysis in one long run-on equation.

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} && (\textit{Definition.}) \\ &= \lim_{x \to a} \frac{x^n - a^n}{x - a} && (\textit{Function evaluation.}) \\ &= \lim_{x \to a} \frac{(x-a)\cdot(x^{n-1} + x^{n-2}\cdot a + \ldots + x\cdot a^{n-2} + a^{n-1})}{x - a} && (\textit{Algebra}) \\ &= \lim_{x \to a} (x^{n-1} + x^{n-2}\cdot a + \ldots + x\cdot a^{n-2} + a^{n-1}) && (\textit{Algebra.}) \\ f'(a) &= na^{n-1}. && (\textit{Thinking - analysis.}) \end{aligned}$$

As usual, in the last equation we replace "a" by "x" to obtain the result claimed by the theorem's statement, $\boldsymbol{f'(x) = nx^{-1}}$.

E.O.P.

Remarks: 1. This proposition allows us to find the derivative for $f(x) = x^{86}$ without difficulty by bringing the exponent "86" **down** in front of the variable x and **reducing the exponent by one** to "85", so that $f'(x) = 86x^{85}$. This helps explain why I call this the "**down-one" rule.** [Actually this is how my first calculus teacher described the rule to me.]

In Leibniz and operator notations this work would be written as follows:

$$\frac{d}{dx}(x^{86}) = \mathbf{86 \cdot x^{86-1}} = \mathbf{86 \cdot x^{85}}$$

$$D_x(x^{86}) = \mathbf{86 \cdot x^{86-1}} = \mathbf{86 \cdot x^{85}}.$$

2. The tangent line interpretation of the derivative helps make some sense of this result by considering the shape of the graphs of power functions of the type x^n. See Figures *** and ***. When n is odd (like 5), n-1 (which would be 4) is even, so the derivative nx^{n-1} will never be negative. The tangent lines to these graphs would all appear to have non negative slopes. When n is even (like 4), n-1 (which would be 3) is odd so the derivative nx^{n-1} will be positive when x is positive and negative when x is negative. The tangent lines to these graphs would all appear to have positive slopes for the curve to the right of the Y-axis while their slopes would be negative to the left of the Y-axis.. Notice also that **the X-axis**, which has a slope of 0, appears to be tangent to these curves at the origin.

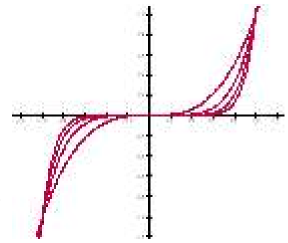

Figure 1
Odd powers of x

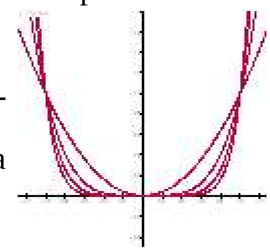

Figure 2
Even powers of x

We generalize this result first to **negative integers** in the next proposition, stated in a form that recognizes that a negative integer is the opposite of a positive integer.

Proposition: I.F.3. (The Power or the "Down One" Rule for Negative Integers.)
For n a positive integer, **if** $f(x) = x^{-n} = \frac{1}{x^n}$ **[$x \neq 0$], then** $f'(x) = -nx^{-n-1} = -\frac{n}{x^{n+1}}$**.**

Notice the proposition expresses the rule using the fact that a negative integer is the opposite of a positive integer. We do this to help avoid some confusion with notation.

Proof: Here again we use the four step analysis along with some algebra to make the argument easier to follow.

Step 1: For n>1

$$\begin{array}{rcl} f(x) & = & x^{-n} \text{ and} \\ - \quad f(a) & = & a^{-n} \\ \hline \end{array}$$

Step 2: $f(x) - f(a) = x^{-n} - a^{-n} = \frac{1}{x^n} - \frac{1}{a^n} = \frac{a^n - x^n}{x^n \cdot a^n} = -\frac{x^n - a^n}{x^n \cdot a^n}$

Inserting this into the difference quotient for f and re-using some of the algebra from the last proof we arrive at

Step 3:
$$\frac{f(x) - f(a)}{x - a} = \frac{x^{-n} - a^{-n}}{x - a}$$
$$= \frac{-(x^n - a^n)}{x^n a^n (x - a)}$$
$$= \frac{-(x^{n-1} + x^{n-2} \cdot a + \ldots + x \cdot a^{n-2} + a^{n-1})}{x^n a^n}.$$

Step 4: Think! So now it's time to think again as we analyze this difference quotient as $x \to a$. The thinking should be familiar since the numerator is certainly approaching $-na^{n-1}$ while the denominator is approaching a^{2n}.

Thus the quotient is approaching $\frac{-na^{n-1}}{a^{2n}} = -na^{-n-1}$, so $f'(a) = -na^{-n-1} = -\frac{n}{a^{n+1}}$.

EOP

Remarks: 1.Using this proposition we see that if $f(x) = 1/x^{86} = x^{-86}$, then the derivative is found simply by bringing the exponent "-86" **down** in front of the variable x and **reducing the exponent by one** to "-87", so that $f'(x) = -86x^{-87} = -86/x^{87}$. Again using Leibniz or operator notation for this example the work would be written as follows:

$$\frac{d}{dx}\left(\frac{1}{x^{86}}\right) = \frac{d}{dx}(x^{-86}) = -86 \cdot x^{-86-1} = -86 \cdot x^{-87} = \frac{-86}{x^{87}}$$

$$D_x\left(\frac{1}{x^{86}}\right) = D_x(x^{-86}) = -86 \cdot x^{-86-1} = -86 \cdot x^{-87} = \frac{-86}{x^{87}}$$

2. The tangent line interpretation of the derivative can also help make sense of this result. Using some graphing technology you might graph x^{-n} for $n = 1, 2, 3,$ and 4 to see for yourself how the derivative information is consistent with the apparent slopes of tangent lines to these graphs.

Fractional Exponents: Once more we generalize the power rule, this time for functions that use fractional exponents of the form $\frac{1}{n}$ where n is a positive integer.

Proposition: I.F.4. (The Power or the "Down One" Rule for $1/n$.)

For ***n* a positive integer**, **if** $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ **and** $x \neq 0$**, then** $f'(x) = \frac{1}{n} x^{(\frac{1}{n}-1)} = \frac{1}{n} (\sqrt[n]{x})^{1-n}$.

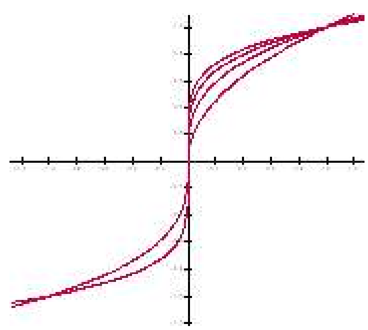

Figure 3
$f(x) = x^{1/n}$

Proof: Another algebraic idea helps with this analysis. As usual we'll find $f'(a)$ and here we assume $a \neq 0$. [See Figure *** for the sketch of some example graphs. Can you see from the graphical interpretation why we might want to exclude the case when $a = 0$?] The idea is simply to substitute y for $f(x)$ and b for $f(a)$.

The definition of the nth root gives us that if $y = \sqrt[n]{x}$ then $y^n = x$ and similarly if $b = \sqrt[n]{a}$ then $b^n = a$. It should make sense from your experience with powers and roots that when x approaches a, y approaches b. (This last statement should be understood intuitively at this stage.) Now we use some of the algebra developed in the previous proofs to observe that

$$x - a = y^n - b^n = (y - b) \cdot (y^{n-1} + y^{n-2} \cdot b + \ldots + y \cdot b^{n-2} + b^{n-1}).$$

Now let's consider the difference quotient, starting with "step 3",

$$\frac{f(x)-f(a)}{x-a} = \frac{y - b}{x - a}$$
$$= \frac{y - b}{(y - b)(y^{n-1}+y^{n-2}b+...+yb^{n-2}+b^{n-1})}.$$
$$= \frac{1}{y^{n-1}+y^{n-2}b+...+yb^{n-2}+b^{n-1}}.$$

Step 4: It's again time to think about the limit, this time as $y \to b$. The denominator has n terms each approaching b^{n-1}, so we see that $f'(a) = \frac{1}{n \cdot b^{n-1}} = \frac{1}{n} \cdot \frac{1}{(a^{1/n})^{n-1}} = \frac{1}{n} \cdot a^{\frac{1}{n}-1}$ **EOP.**

Remarks: 1. Using this proposition we see that if $f(x) = x^{1/86}$, then the derivative is found easily again by bringing the exponent "1/86" **down** in front of the variable x and **reducing** the exponent by **one** to "-85/86", so that $f'(x) = (1/86)x^{-85/86}$. Again using Leibniz or operator notation for this example would be written as follows:

$$\frac{d\ (\sqrt[86]{x})}{dx} = \frac{d(x^{\frac{1}{86}})}{dx} = \frac{\mathbf{1}}{\mathbf{86}} \cdot x^{\frac{1}{86}-1} = \frac{\mathbf{1}}{\mathbf{86}} \cdot x^{-\frac{85}{86}}$$

$$D_x(\sqrt[86]{x}) = D_x(x^{\frac{1}{86}}) = \frac{\mathbf{1}}{\mathbf{86}} \cdot x^{\frac{1}{86}-1} = \frac{\mathbf{1}}{\mathbf{86}} \cdot x^{-\frac{85}{86}}$$

2. The tangent line interpretation applied to this derivative is worth some thought for exponents $1/n$ when n is an odd number. Can you see why the slopes of tangent lines will always be positive as might seem to make sense from Figure ***?

With the previous propositions as models for how to combine algebra with the definition of the derivative to find these derivatives, the generalization of the power rule to positive and/or negative fractional exponents is left as an exercises for the reader. [See exercises ***]

(Hint: Substitute $y = x^{1/n}$ as in the previous proposition.) We will return to these generalizations in later sections after we have developed more rules for the calculus of derivatives.)

I.F.1*b* Derivatives and Linearity: Adding and changing scales of values.

Preface: Two of the most common ways that we combine numbers and variables is by **addition and multiplication**. Starting with positive integer powers of a variable and constants we can multiply and add to form **polynomial functions** like $p(x) = 5+3x-2x^2+7x^3$ and $q(t) = 3.4-\pi t-\frac{3}{7}t^2$.

Of course by using other powers more complicated elementary functions can be formed using addition and constant multiplication. The calculus for derivatives provides two very useful and easy rules for finding the derivatives of these functions.

First we will investigate the effect of adding and scalar multiplication on the derivatives of functions by considering linear functions. Their derivatives are simple to compute. Linear functions also provide a pattern for what is happening with other functions because the derivative is closely related through the geometric interpretation to the slope of the tangent line and the motion interpretation at a constant rate.

Example I.F.1. Let $f(x) = 5x + 3$ and $g(x) = 4x - 2$.
Suppose that $s(x) = f(x) + g(x)$ and $k(x) = 12 f(x)$.
Find $s'(x)$ and $k'(x)$ and compare these to $f'(x)$ and $g'(x)$.
Solution: With such simple functions we can easily do the arithmetic:
$s(x) = f(x) + g(x) = (5x + 3) + (4x - 2) = 9x + 1$.
So $s'(x) = 9$.

$k(x) = 12(5x + 3) = 60x + 36$.
So $k'(x) = 60$.
Now since $f'(x) = 5$ and $g'(x) = 4$ our work shows that $\mathbf{s'(x) = f'(x) + g'(x)}$ **and** $\mathbf{k'(x) = 12 f'(x)}$**.**

Before working through an argument for the general result about derivatives for sums and scalar multiples of functions, let's work through one not so trivial example to see how the definition interacts with the algebra.

Example I.F.2. Let $f(x) = x^2$ and $g(x) = x^5$. Suppose that $s(x) = f(x) + g(x)$ and $k(x) = 12 f(x)$. We will find $s'(x)$ and $k'(x)$ and compare these to $f'(x)$ and $g'(x)$.

Exploration: As you might expect, we'll use the four step approach with the derivative definition that considers the limit with $h \to 0$, $s'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}$. This allows us to use the variable x as it appears in the function characterization of the derivative by using h to control the estimations.

Step 1: $s(x+h) = f(x+h) + g(x+h) = (x+h)^2 + (x+h)^5$ and
$-\quad s(x) = f(x) + g(x) = x^2 + x^5$.

Step 2: $s(x+h) - s(x) = (x+h)^2 - x^2 + (x+h)^5 - x^5$.

Step 3: $\frac{s(x+h) - s(x)}{h} = \frac{(x+h)^2 - x^2 + (x+h)^5 - x^5}{h} = \frac{(x+h)^2 - x^2}{h} + \frac{(x+h)^5 - x^5}{h}$

Notice that the summands in the last expression are precisely the difference quotients used to find the derivatives of the functions $f(x) = x^2$ and $g(x) = x^5$. Since we know the derivatives of these functions from our recent work on powers of x, we will not proceed any further with the algebra.

Step 4: Think! Now it's **time to think** about this difference quotient as $h \to 0$.

Analyzing the limit of the difference quotient as $h \to 0$ we see that $\frac{(x+h)^2 - x^2}{h} \to 2x$ and $\frac{(x+h)^5 - x^5}{h} \to 5x^4$. So $\frac{s(x+h) - s(x)}{h} = \frac{(x+h)^2 - x^2}{h} + \frac{(x+h)^5 - x^5}{h} \to 2x + 5x^4$.

We conclude that $s'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h} = 2x + 5x^4$.

Notice that we have found again that $s'(x) = f'(x) + g'(x)$.

For the function k we follow the same ideas:

Step 1: $k(x+h) = 12 f(x+h) = 12 (x+h)^2$ and

$-\quad k(x) = 12\ f(x) \quad = 12\ x^2$.

Step 2: $k(x+h) - k(x) = 12 (x+h)^2 - 12 x^2 = 12[(x+h)^2 - x^2]$.

Step 3: $\frac{k(x+h)-k(x)}{h} = \frac{12[(x+h)^2-x^2]}{h} = 12\frac{(x+h)^2-x^2}{h}$

Step 4: Think! When we analyze the difference quotient as $h \to 0$, we have $\frac{(x+h)^2-x^2}{h} \to 2x$ and we conclude that $k'(x) = \lim_{h \to 0} \frac{k(x+h)-k(x)}{h} = 12 \cdot 2x = 24x$.

We notice that $k'(x) = 12 f'(x)$.

These examples suggest the general result about the derivatives of sums and scalar multiples of functions which we now prove using the definition of the derivative. These rules are called the **Sum Rule and the Scalar or Constant Multiple Rule**. We will refer to them jointly sometimes as the **linearity properties of the derivative**.

Theorem I.F.5. (Linearity of the Derivative.) Suppose f and g are differentiable at x and that $\mathbf{s(x) = f(x) + g(x)}$ and $\mathbf{k(x) = \alpha f(x)}$ where α is a constant real number. Then

i). $\mathbf{s'(x) = f'(x) + g'(x)}$ **(The Sum Rule)**

and ii). $\mathbf{k'(x) = \alpha f'(x)}$ **. (The Scalar or Constant Multiple Rule.)**

Proof: The proofs are not to hard to follow provided you keep in mind the concept that they merely eliminate the use of the specific functions in the last example.

We'll again use the four step analysis for the derivative that considers $h \to 0$.

Step 1: $s(x+h) = f(x+h) + g(x+h)$ and

$-\quad s(x) = f(x) + g(x)$

Step 2: $s(x+h) - s(x) = f(x+h) - f(x) + g(x+h) - g(x)$.

Step 3: $\frac{s(x+h)-s(x)}{h} = \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} = \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$

Step 4: Think! Now it's **time to think** about this difference quotient as $h \to 0$.

Notice that the summands in the last expression are precisely the difference quotients used to find the derivatives of the functions $f(x)$ and $g(x)$. We have assumed that f and g have derivatives at x, so as $h \to 0$, **the difference quotients for these functions must approach their derivatives, i.e.**, as $h \to 0$, $\frac{f(x+h)-f(x)}{h} \to f'(x)$ and $\frac{g(x+h)-g(x)}{h} \to g'(x)$.

So $\frac{s(x+h)-s(x)}{h} = \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \to f'(x) + g'(x)$.

Forthe function k we follow the same ideas:

Step 1: $k(x+h) = \alpha\ f(x+h)$ and

$-\quad \underline{k(x) = \alpha\ f(x)\quad .}$

Step 2: $k(x+h) - k(x) = \alpha\, f(x+h) - \alpha\, f(x) = \alpha\, [f(x+h) - f(x)]$.

Step 3: $\frac{k(x+h)-k(x)}{h} = \frac{\alpha f[(x+h)-\alpha f(x)]}{h} = \alpha\ \frac{f(x+h)-f(x)}{h}$

Step 4: Think! When we analyze the difference quotient as $h \to 0$, as before we have $\frac{f(x+h)-f(x)}{h} \to f'(x)$ and we conclude that $k'(x) = \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} = \alpha \cdot f'(x)$. **EOP.**

Notation: Suppose $y = f(x)$ and $z = g(x)$. The following formulae express the linearity properties for functions in Leibniz and operator notations:

Leibniz	**Operator**
$\frac{d}{dx}(y+z) = \frac{d}{dx}y + \frac{d}{dx}z$;	$\mathbf{D}_x(y + z) = \mathbf{D}_x\, y + \mathbf{D}_x\, z$ $\mathbf{D}(f(x) + g(x)) = \mathbf{D}(f(x)) + \mathbf{D}(g(x))$;
$\frac{d}{dx}(\alpha y) = \alpha \frac{d}{dx} y$.	$\mathbf{D}_x(\alpha\, y) = \alpha\, \mathbf{D}_x\, y$ $\mathbf{D}(\alpha\, f(x)) = \alpha\, \mathbf{D}f(x)$.

Comment: Notice that the sum rule essentially distributes the derivative symbols to the two summands while the scalar multiple rule allows the derivative symbols to move past the scalar to operate on the function factor alone.

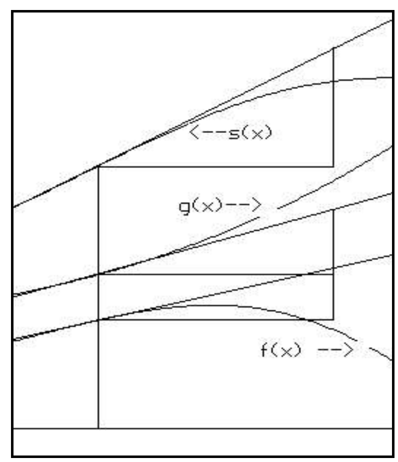

Figure 4
$S(x) = f(x) + g(x)$

Interpretations: a) Graphical. The graph of the sum, s, results from placing the graph of g with horizontal axis lying on top of the graph of f. See Figure ***. This will shift the inclination of the tangent line to g at x in the resulting curve s so that its slope is $f'(x) + g'(x)$.

The scalar multiple can be thought of as merely changing the scales on the Y-axis by the scalar α. The tangent line to the curve will be the same on the graph but it's slope will be changed by the factor of α because of the change in the vertical scale.

b) Motion. To think of the function s in terms of a motion interpretation, imagine a long train moving on a line with the distance of its caboose from the end of the terminal platform measured by f(t) meters at time t minutes. You are on the train walking on the same line with your distance from the end of the caboose measured by g(t) meters at time t minutes.

Thus your position is *s*(t) meters from the end of the terminal platform at time t minutes. Your velocity with respect to the caboose is given by *g*'(t), but your velocity with respect to the terminal platform is *s*'(t) = *f*'(t) + *g*'(t) since the velocity of the train must be added to your velocity to account for the train's movement.

Multiplication by a constant can be visualized with a transformation figure in which any segment on the source line is magnified (or reduced) by a constant factor, the scaling factor, on the target line. See Figure***.

To interpret the scalar multiplication rule in a motion context we need only consider that we are changing the scales on the line for a moving object. So if the object is travelling on a line with its scale measured in feet and we change the scale to inches then we would multiply the positions *f*(t) by 12 to consider *k*(t)=12 *f*(t). The velocity at time t would be multiplied by 12 as well to give a consistent measure of the velocity at time t, so *k*'(t) = 12 *f*'(t).This can be visualized with a transformation figure showing the magnification of scales after the position is located first on a scale measured in feet, then transformed to a scale measured in inches.See Figure ***

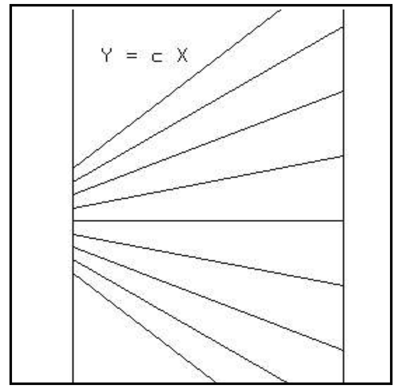

Figure 5

c) Economics. For addition, consider *C*(*x*), the cost of producing *x* units of a commodity (such as a yard of linen fabric or a ton of cement mix) being made up of the cost of materials *f*(*x*) dollars and the cost of labor *g*(*x*) dollars. Thus, *C*(*x*)=*f*(*x*) +*g*(*x*). The marginal change in total production cost for an additional unit of production *C*'(*x*) would be determined by the sum of the marginal change in the costs of materials, *f* '(*x*), and the marginal change in the costs of labor, *g*'(*x*).

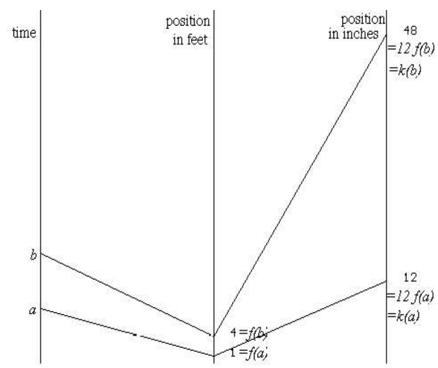

Figure 6

For an economic interpretation of scalar multiplication, suppose the county tax rate on the selling price of a commodity is 6%. This means that the daily county tax revenues, *R*, (from which the county will pay for the general services it provides) is .06 *S*, where *S* is the dollar value of the total daily sales revenues for the county. If the daily sales revenues on June 15th are increasing at a rate of \$5000 per day, we can find the rate at which the tax revenues are increasing since they are a scalar multiple of the sales revenues. In other words we can find the derivative of *R* as a function of time since we know the derivative of *S* as a function of time is 5000. See Figure ***. Here is the way this might be expressed in Leibniz notation: $\frac{d}{dt}R = \frac{d}{dt}(.06S) = .06\frac{d}{dt}S = .06\frac{\$5000}{day} = \frac{\$300}{day}$.

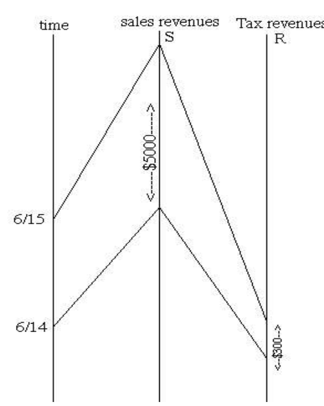

Figure 7

Interpretations of Multiplication: Interpretations of addition are fairly simple because the numbers that we add in any interpretation must represent the same type of measured quantity. [This is sometimes referred to as the **homogeneity** issue.] How often have you heard a teacher say, "You can't add apples and pears... You can add pieces of fruit and more pieces of fruit." This simple but important principle for using numbers is essential for any application using addition. On the other hand multiplication has many different possible interpretations making it a much richer tool for modeling. As we proceed through our studies we will find it useful to consider some of the following interpretations that make sense for multiplication.

Repeated Addition. The first, and perhaps simplest, interpretation of multiplication is that multiplication is repeated addition. When we place 5 lengths of measurement 12 meters together in succession on a line and ask for the total measurement we multiply 5 times 12 to conclude that the total measurement is 60 meters. With this interpretation one of the numbers measures some quantity specific to the application while the other is a measure of (abstract) numerical units. This interpretation is rather limited since to make sense one of the numbers must be a counting number or an integer.

Scale Change. A second interpretation is that multiplication is changing a scale. When we change the measurement of 5 feet to inches we multiply 5 *feet* by $12\frac{inches}{feet}$ (the scale conversion factor measured as inches per foot) to conclude the measurement in inches is 60 inches. In this interpretation one number measures a quantity specific to the application while the other number, usually a constant, gives the scale conversion factor as a ratio.

Rates. A third interpretation of multiplication uses a product to find the change in a measured quantity which is changing at a fixed rate. We use this view when we find the distance traveled in 5 seconds when a person runs at a fixed rate of $12\frac{feet}{second}$. The distance traveled is the product, 5 *seconds* times $12\frac{feet}{second}$ = 60 *feet*. Likewise we would employ this interpretation to find that the total price for purchasing 5 liters of champagne at a price of a $12 per liter is 5 times 12 = $60. In this product one number measures a rate of change or exchange while the other number gives a measure of a variable controlling the change.

Proportion. Finally multiplication can be interpreted as relating different concepts that are connected quantitatively by direct proportions. For example the area of a rectangle with a fixed base length is proportional to the length of its width. As a result to determine the area of a rectangle 5 feet wide and 12 feet wide we multiply 5 *feet* by 12 *feet* so the area is 60 *square feet*. More subtly perhaps, we consider the work done by a group of people proportional to the number of people working as well as to the amount of time spent on the task. So when we measure the work done by 5 *people* each working 12 *hours* we multiply to find they have worked together 60 *person-hours*. This interpretation is found as well in measuring the work done by applying a force to move an object through a distance, such as lifting a heavy box. For example, it is common to measure the amount of work done by lifting 12 *pounds* up 5 *feet* by multiplying to give the measure as 60 *foot-pounds*.

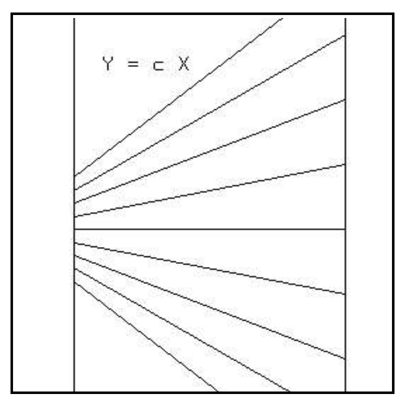

Figure 8

Since multiplication has such a variety of interpretations it is helpful initially to focus on one interpretation. We will use that of changing scale here and presume that the change of scale factor doesn't vary- so we are thinking of multiplication of each measured quantity by the same number, the scalar factor. Thus we call multiplication by a constant, scalar multiplication. [Other ways to think of this type of multiplication be a constant would be using a fixed rate, velocity or unit price, or a fixed length of a rectangle with varing width in an area problem. You should construct your own examples of interpretations related to constant multiplication.]
Multiplication by a constant can be visualized with a transformation figure in which any segment of a given measurement is magnified (or reduced) by a constant factor, the scaling factor. See Figure ****

Formal Application of the Linearity Properties: The linearity properties of the derivative are quite simple to use after a little experience. With them and the derivatives of the power functions found in the first part of this section, we can find the derivatives of polynomial functions as well as fairly complicated combinations of functions without resorting to the definition of the derivative and the complications of finding limits.

Example I.F.3. Find the following derivatives as indicated:

a) Find $D_x s(1)$ when $s(x) = 7x^3 + \frac{6}{x}$.

b) Find $\frac{dy}{dx}$ when $y = \frac{3+5x^6}{\sqrt{x}}$

Solutions: a) We start by recognizing that the last operation used to find the value of s is addition. Ignoring the constant factors of 7 and 6 in these two summands we let $f(x) = x^3$ and $g(x) = 1/x$. Then we can reassemble the function s,

$$s(x) = 7\,f(x) + 6\,g(x).$$

Using the Sum Rule we have

$D_x s(x) = D_x\,(7\,f(x)) + 6\,g(x)) = D_x\,(7\,f(x)) + D_x\,(6\,g(x))$.

Now we can use the scalar multiple rule to obtain the derivatives of the individual summands so that $D_x(7f(x)) = 7D_x\,f(x)$ and $D_x\,(6g(x)) = 6D_x\,g(x)$ therefore $D_x s(x) = 7D_x f(x) + 6D_x g(x)$. Replacing these derivatives with the appropriate values from the power rules gives the result, $D_x s(x) = 7\,(3x^2) + 6(-1/x^2)$ and thus $D_x s(1) = 21 - 6 = 15$.

This work to find $D_x s(x)$ can done more efficiently by abusing the notation of the expressions involved. This allows us to avoid naming the functions as we apply the sum and the scalar multiple rules. Here's how this work might be written with this slight abuse of notation.

$D_x s(x)$	$= D_x(7x^3 + 6/x)$	
	$= D_x(7x^3) + D_x(6/x)$	(Sum Rule)
	$= 7\,D_x(x^3) + 6\,D_x(1/x)$	(Scalar Multiplication Rule)
	$= 7\,(3x^2) + 6\,(-1/x^2)$	(Power Rule)
	$= 21\,x^2 - 6/x^2$.	

At this stage replace the "x" in the expressions with the number 1 and you have the same result, $D_x s(1) = 21\,(1)^2 - 6/(1)^2 = 15$.

b) Using the same style of notational abuse we find dy/dx. First we recognize that as the function is currently expressed as a rule, the last operation used to compute the value of y is division. To allow us to use the sum rule we must use a little algebra to express the function y with a procedure that concludes with addition. This is not too hard once we recognize we can perform the division: $y = \frac{3+5x^6}{\sqrt{x}} = 3x^{-\frac{1}{2}} + 5x^{\frac{11}{2}}$. We continue now very similarly to part a), so

$$\frac{d}{dx}y = \frac{d}{dx}(3x^{-\frac{1}{2}} + 5x^{\frac{11}{2}})$$

$$= \frac{d}{dx}(3x^{-\frac{1}{2}}) + \frac{d}{dx}(5x^{\frac{11}{2}}) \qquad \text{(Sum Rule)}$$

$= 3\frac{d}{dx}(x^{-\frac{1}{2}})+5\frac{d}{dx}(x^{\frac{11}{2}})$ (Scalar Multiplication Rule)

$= 3(-\frac{1}{2})\,x^{-\frac{3}{2}}+5(\frac{11}{2})\,x^{\frac{9}{2}}$ (Power Rule)

$= \frac{-3}{2}\,x^{-\frac{3}{2}}+\frac{55}{2}\,x^{\frac{9}{2}}.$

Exercises I.F.1

1. Find $f\,'(1)$ for the following functions:
   a. $f(x) = x^{36}$ b. $f(x) = 1/x^{36}$ c. $f(x) = x^{1/36}$
2. Find $f\,'(1)$ and $f\,'(-1)$ when
   a. $f(x)= x^2 + 6x - 3$ b. $f(t)= t^3 - 5t^2 + t$
   c. $f(u)= u^{20} - 4u^5$ d. $f(z)= 1 + z + 2z^2 + 6z^3 + 24z^4$.
3. Find $f\,'(1)$ when $f(x) =$
   a. $3x^5+2x^3$ b. $2\sqrt{x}+\frac{3}{x^5}$
   b. $(x^3+2x^2)\cdot x^{\frac{1}{3}}$ d. $\frac{x^4+2x}{x^2}$
4. Find $D_t f(t)$ and $D_t f(4)$ when $f(t) =$
   a. $-16t^2+15t+2$ b. $2\sqrt{t}+4(\sqrt{t})^5$
   c. $\frac{t^2+1}{t}$ d. $\frac{2}{3}t^{\frac{3}{2}}+\frac{3}{2}t^{\frac{2}{3}}$
5. Find $\frac{d}{dx}y$ when $y =$
   a. $x^5+x^4+x^3+x^2+x+1$ b. $1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4$
   c. $\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}$ d. $(3+x)^2$
6. **The derivative of any polynomial**:
   Find $P\,'(x)$ when $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.

Generalize your result to give a formula for the derivative of any polynomial function,

$P(x)=a_0+a_1x+a_2x^2+...+a_{n-1}x^{n-1}+a_nx^n$ where $a_0,a_1,a_2,...,a_{n-1},a_n$ are all constants.

7. Suppose an object is s(t) feet above ground level at time t seconds where $s(t) = -16t^2 + 64t$ where $t \geq 0$. Find when the velocity of the object is positive . When it the velocity negative? Interpret these answers in terms of the motion of the object. When is the velocity of the object zero? What is the highest the object was above ground level?

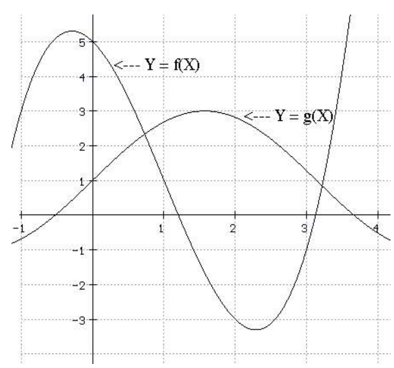

Figure 8

8. Suppose the functions f and g have graphs as in Figure ***. Based on these graphs estimate the following derivatives:
   a. $s'(0)$; $s'(1)$; $s'(2)$ where $s(x) = f(x) + g(x)$.
   b. $k'(0)$; $k'(1)$; $k'(2)$ where $k(x) = 5f(x)$.
   c. $p'(0)$; $p'(1)$; $p'(2)$ where $p(x) = 2f(x) + 5g(x)$.
9. Find all x where $f'(x) = 0$ when
   a. $f(x) = \text{A}x^2 + \text{B}x + \text{C}$ b. $f(x) = x^3 + ax^2 + bx + c$ c. $f(x) = x + 1/x \quad x \neq 0$.
10. For $x > 0$, draw a graph of $y = x^2$ and $y = x^{1/2}$. Draw the line tangent to the first graph at (2,4). Draw the line to the second graph at (4,2). How are the slopes of these lines related numerically? Notice that these graphs are symmetric with respect to the line $y=x$. Use this symmetry to explain the numerical relation of the slopes of the tangent lines.
11. Draw a graph of $y = x^3$ and $y = x^{1/3}$. Draw the line tangent to the first graph at (2,8). Draw the line to the second graph at (8,2). How are the slopes of these lines related numerically? Notice that these graphs are symmetric with respect to the line $y=x$. Use this symmetry to explain the numerical relation of the slopes of the tangent lines.

Reversing the process: Once we are able to find derivatives using the representation of a function algebraically we can begin to explore the issues of finding algebraic expressions describing functions based on information about derivatives. The next problems ask you to consider some of these.

12. The position of an object moving on a straight line at t second is s(t) meters to the right of a given point, and $s(t) = 1 + kt^2$.The velocity is observed at 3 seconds to be 12 meters per second. a. Find k. b. Find s(5).
    c. At what time(s) was the object 201 feet from the object?
13. Find a function P where $P'(x) = f(x)$ for the following:
    a. $f(x)=3$ b. $f(x)=6x + 3$ c. $f(x)=9x^2 +6x +3$ d. $f(x)= \text{A}x^2 + \text{B}x + \text{C}$.
14. Suppose $f(x)=Ax^2+Bx+C$. Find A, B, and C so that $f(0)= 4$, $f'(0) = 2$ and $f'(1)=6$.
15. For each of the following find a function $y(x)$ so that
    a. $\frac{d}{dx}y = 3x^2+4x^3+5x^4$; $\frac{d}{dx}y = 1+x+x^2+x^3$;
    b. $\frac{d}{dx}y = \frac{1}{x^2}+\frac{1}{x^3}$; $\frac{d}{dx}y = 3x^3-5x^{1/3}+\frac{2}{\sqrt{x}}$.
16. Find a function f where $D_x f(x) = 6x^2+4x-3$ and $f(1) = 2$.
17. The Difference Rule: $\mathbf{D}(f(x) - g(x)) = \mathbf{D}(f(x)) - \mathbf{D}(g(x))$. a.Use the Linearity Properties to justify the Difference Rule. Restate the rule in Leibniz and function notations.
    b. Use the definition of the derivative to justify the Difference Rule.
18. Follow the proof of Proposition I.F.3 to show that if $f(x) = 1/x^5$ then $f'(x)=-5/x^6$.
19. Follow the proof of Proposition I.F.4 to show that if $f(x) = x^{1/5}$ the $D_x f(x) = 1/(5x^{4/5})$.
20. Using the definition of the derivative, show that $\frac{d}{dx}(x^{\frac{3}{5}}) = \frac{3}{5}x^{-\frac{2}{5}}$
21. Prove the power rule for exponents that are rational numbers.
22. Suppose $F(x)$ is a probability distribution for random variable x on the interval [0, 2]. Find the probability density at x=1 when $F(x)$ is as follows:

a. $F(x) = 1/4\, x + 1/8\, x^2$ b. $F(x) = 1/8\, x + 3/16\, x^2$

c. $F(x) = a\, x + b\, x^2$ where a>0, b>0, and 2a + 4b = 1.

d. $F(x) = 1/8\, x^2 + 1/16\, x^3$. e. $F(x) = 1/6\, x + 1/12\, x^2 + 1/24\, x^3$.

23. We say that two curves are **orthogonal** at the point (a,b) if the curves pass through the point (a,b) and the tangent lines to the curves at (a,b) are perpendicular.
    a. Show that the graphs of $f(x) = x^2$ and $g(x) = 1/\sqrt{x}$ are orthogonal at the point (1,1).
    b. Find p so the graph of g where $g(x) = x^p$ is orthogonal to the graph of f where $f(x) = x^{10}$ at (1,1).
    c. Give a general statement on when the graph of g where $g(x) = x^p$ will be orthogonal to the graph of f where $f(x) = x^q$. Prove it.
    d. Are the curves with equations $y = x^2$ and $y = 1 - x^2$ orthogonal?
    e. Find any A where the graph of $y = A - x^2$ is orthogonal to the graph of $y = x^2$.
24. Research/reading project: Look in some other calculus books to find a different proof of the derivative of the power functions using the binomial theorem. Discuss the advantages and disadvantages of using the binomial theorem for the proof.