I. E. Notation and Terminology: the Good, the Bad, and the Useful

I.E.1 The language of approximation and estimation: LIMITS.

Throughout our discussion so far we have estimated and approximated many numbers using the arrow, →, to indicate this estimation in the notation. For example, we have written that **as $x \to 3$, $x + 6 \to 9$** intending to convey the understanding that **when x is a number close, but not equal, to 3, the related number x+6 is close to 9.**

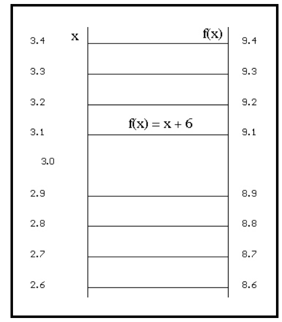

Figure 1

It is common in mathematics to express this complex relationship using the word **"limit."** Don't let the word frighten you...in its simplest use it merely describes a number.

In the case of the statement that $x+6 \to 9$, **the number 9 is the limit**.

Other uses of the word "limit": In geometry one might describe a point on the boundary of a region as a limit point for that region. Or when you walk toward a wall you might say that the wall was the limit of your motion. When you fill a glass to the top we often say that you had filled it to its limit. And we talk about the limit of a biological population size that can be supported in a particular environment. So, in the context of numbers and functions a limit describes a number (here 9).

The limit number is estimated by other numbers described by an expression (here x+6) with a controlling variable (here x) chosen close to a specific number (here 3). As you think of approaching a wall, but possibly never reaching it, so the numerical values of the expression x+6 are approaching the number 9. Traditional ways of expressing the limit relationship are with phrases such as "the limit of $x + 6$ as x approaches 3 is 9," "the limit as x approaches 3 of $x + 6$ is 9," or "9 is the limit of x+6 as x approaches 3".

Similarly, as another example, we translate the symbolic phrases that as $x \to 5$, $2x + 1 \to 11$ by saying that as x approaches 5, the corresponding number $2x + 1$ approaches the limit of 11.

For hundreds of years mathematicians struggled to give a precise mathematical meaning to the intuitively apparent use of the concept of limit. So you should not feel badly if the concept of a limit number seems a little vague at this stage. You will find a precise and rigorous mathematical treatment of this concept in the Appendix for this chapter. At some time before you finish a course in calculus you should try to comprehend this definition. For now let's assume that you are developing a working intuition of when a number is a limit of an expression as well as some thought patterns about these situations that help you find a limit.

Keep in mind that finding the limit of the difference quotient is a **key part** of the process of finding the derivative of a function. Remember: It is the difference quotient $\frac{f(x)-f(a)}{x-a}$ that is the focus of attention for this process, **not** the function f.

In the language of limits, the derivative of f at a, $f\,'$(a), is the limit of the difference quotient, $\frac{f(x)-f(a)}{x-a}$, as x approaches a (or as $x \to a$), provided the limit exists.

I.E.2. The Notation of limits. In the attempt to use symbols to communicate concepts, mathematicians have developed a standard notation, evolved over hundreds of years along with the concepts of the derivative and limits. To express the fact that 9 is the limit of $x + 6$ as x approaches 3, we write $\lim_{x \to 3} x + 6 = 9$.

Many historians give the major credit for clarifying the basic concepts to the 19 th century French mathematician Augustin-Louis Cauchy (1789 - 1857). The German mathematician Karl Weierstrass (1815-1897) is usually credited with successfully making rigorous the numerical foundation of limits.

It is also important to be able to comprehend statements written using this notation, so that $\lim_{x \to 5} 2x + 1 = 11$ expresses the fact that as x approaches 5, the corresponding number $2x+1$ approaches the limit number of 11.

The definition of the derivative with the conventional limit notation is written

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ **provided the limit exists**.

Example I.E.1. Suppose $f(x) = x^2 + 3$. Find $f'(2)$ using conventional limit notation.

Solution: Using conventional limit notation with the definition can consolidate the four step process for finding the derivative into one long run-on equation. This may look more concise, but you should still recognize that we are first evaluating the function, simplifying using some algebra, and finally thinking to analyze the simplified expression and find the limit number.

$$\mathbf{f'(2)} = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$ [*The derivative definition using a* = 2.]

$$= \lim_{x \to 2} \frac{(x^2 + 3) - 7}{x - 2}$$ [*Evaluating using the definition of f*.]

$$= \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$ [*Algebra*.]

$$= \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2}$$ [*Algebra*.]

$$= \lim_{x \to 2} x + 2$$ [*More Algebra*.]

$$= \mathbf{4}.$$ [*Thinking*: *What happens when* $x \to a$?]

Notice that the abbreviation "lim" appears in all but the last line. Omitting this symbol might seem convenient, but leads to nonsensical statements. **Without this symbol** we would be saying that an open expression using the variable x without any further qualification was a number, in this case the number 4. If you choose to write your work for finding the derivative with the conventional limit notation BE CAREFUL to write the "lim " whenever appropriate. This usually means writing “lim” until the precise numerical value of the limit number is determined.

I.E.3. Other notation used to express the definition of the derivative.

It is common to express the definition of the derivative in at least two alternative notations which have historical and conceptual significance.

Cauchy's work on the calculus emphasized the importance of the changes examined in finding the derivative by using a special symbol, the Greek letter delta, Δ, together with the coordinate variable name to indicate that change. Thus we replace the change in the controlling variable, $x - a$, by Δx and the change in the controlled variable, $f(x) - f(a)$, by Δy.

In fact with this notation, $x = a + \Delta x$. These symbols are not new to our work, since we have used them in our work on slope of lines and average velocity. Now when $x \to a$ it should make sense that $\Delta x = x - a \to 0$, and conversely, when $\Delta x \to 0$ then $x \to a$. The definition of the derivative of f at a with the Δ notation is given by

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x},$$ **provided the limit exists.**

Replacing a by x in the last equation leads to a sometimes confusing formula for the beginner:

$$f'(x) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

In this form it is important to recognize that the number x *is not related to the number* Δx. The number x can be any size, but **the number Δx should be considered close to 0.**

In the same vein, but with a slightly less confusing notation, we use the letter h to represent the change in the controlling variable, so $\mathbf{x-a = h = \Delta x}$ and thus $x = a + h$.

Now $x \to a$ if and only if $h \to 0$ and the definition of the derivative of f at a is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ or } f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$ **provided the limits exists.**

Example I.E.2. We redo the work in Example I.E.1 using the limit as $h \to 0$ to express the definition of the derivative.

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{((2+h)^2+3)-7}{h} \\ &= \lim_{h\to 0} \frac{4+4h+h^2-4}{h} \\ &= \lim_{h\to 0} \frac{4h+h^2}{h} \\ &= \lim_{h\to 0} 4+h \\ &= 4. \end{aligned}$$

Note that the work in this example is slightly easier than the work in I.E.1. The common factor h can be recognized easily and cancelled without requiring the factoring of $x^2 - 4$. In the thinking part of the work using $h \to 0$ for every problem eliminates the need to adjust the frame of reference for this part of the analysis to the particular number a.

I. E. 4 Other notations for the derivative.

There are several alternatives to the notation $f'(a)$ for the derivative as a number and f' as notation for the function. We frequently denote a function by a variable name, such as y or u, so we might write $y = f(x) = x^2$. Sometimes it is convenient to name the function by its rule of definition, so in this case we might merely name the function x^2 without any more formal name. This can be confusing since **the symbol x^2 is used to both name the function and to give the value of the function for the number *x*.**

Now we want a notation to describe the derivative as a number and also the derivative as a function. **For the number**, think of a as fixed number and the variable x is set equal to a. Then we might write $f(a) = a^2 = y\,|_{x=a}$. So for instance $f(3) = 9 = y\,|_{x=3}$.

Now for the derivative at $x = a$ we have the following variations on the standard $f'(a)$ function notation:

Leibniz	Mixed function/Leibniz	Operator	Mixed function/archaic evaluation
$f'(a) = \frac{dy}{dx}\Big\vert_{x=a}$	$f'(a) = \frac{df}{dx}(a)$	$f'(a) = D_x f(a)$	$f'(a) = y'\,\Big\vert_{x=a}$

Here are several ways to write $f'(3)$ when $f(x) = y = x^2$:

$f'(3) = \frac{dy}{dx}\Big\vert_{x=3}$	$f'(3) = \frac{df}{dx}(3)$	$f'(3) = D_x f(3)$	$f'(3) = y'\,\Big\vert_{x=3}$
$f'(3) = \frac{df}{dx}\Big\vert_{x=3}$		$f'(3) = D_x(x^2)(3)$	$f'(3) = (x^2)'\,\Big\vert_{x=3}$
$f'(3) = \frac{d}{dx}(x^2)\Big\vert_{x=3}$		$f'(3) = D f(3)$	

When the derivative is considered as a function itself, the standard function notations are f' or $f'(x)$. Alternatives are $\frac{dy}{dx}$, $D_x f(x)$, y', $y'(x)$, or $Df(x)$. So, in the example we have been considering we might write $f'(x) = 2x = \frac{d(x^2)}{dx} = D(x^2) = D_x(x^2) = (x^2)'$.

As you can see this is a potentially confusing situation, but **there is no way to change the history of notations**. You'll just have to get used to understanding each of these (and others not mentioned here) as representing the derivative of a function. Each of these notations has some advantage. The Leibniz notation appears to be a quotient which reminds us of the slope of a line or the average velocity of a moving object. However **the derivative is not a quotient**, but is the limit of the difference quotients. So the Leibniz notation also reminds us that the definition of the derivative does involve division. For this reason it is useful in reminding us that the derivative has a variety of interpretations related to ratios of quantities connected various model situations. Ratio also suggests some kind of rate may be involved in the interpretation of the derivative. In fact the notation $\frac{dy}{dx}$ is often read as "the derivative of y with respect to x" with an alternative reading as "the instantaneous rate of change of y with respect to x."

The function notation (due to the French mathematician Joseph Lagrange, 1736-1813) is the most abstract, easiest to write, and easiest to miss in reading a statement. The notation provides explicitly for the essentials: the name of the original (or primitive) function, f; a way to recognize that it is the derivative function, the prime '; and a way to specify the number or variable at which the derivative is to be evaluated.

The operator notation has the benefit of using the D of "derivative" and being able to specify the variable in case there is some ambiguity in the function being studied. The operator notation thus has some of the advantages of the Leibniz notation while being about as easy to write as the function notation.

Exercises I.E

1. Write the following statements using the limit notation:

a. As $x \to 3$, $3x + 2 \to 11$.
b. As $t \to 2$, $1/t \to 1/2$.
c. As $u \to -4$, $u^2 - 4u + 1 \to 1$.
d. As $y \to -1$, $y/(y^2 + 1) \to -1/2$.

2. Find the following limits:

a. $\lim_{x\to 1} 3x - 5$ b. $\lim_{x\to 1} x^2 - 2$ c. $\lim_{x\to -5} 1/(x + 3)$ d. $\lim_{x\to 27} 7$

3. There are advantages and disadvantages to each of the approaches to finding the derivatives depending on the controlling variable used in the limit notation. After completing this exercise write a brief comparison of what you find easier or harder with using each notation.

For each of the functions below find the derivatives as indicated:

i. Using the definition with x as the controlling variable for the limit.
ii. Using the Δx as the controlling variable for the limit.
iii. Using h as the controlling variable for the limit.

a. $f(t) = t^2 - 3t + 1$; Find $f'(2)$ and $f'(x)$.
b. $f(x) = 2x^2 - 2x - 3$; Find $f'(2)$ and $f'(x)$.
c. $f(u) = u^3 - 3u^2 + 1$; Find $f'(2)$ and $f'(x)$.
d. $f(t) = 2x^3 - 2x^2 -5$; Find $f'(2)$ and $f'(x)$.

4. A common error in notation comes with a confusion of the parenthesis used for functions. This error is often made in the use h or Δx as the controlling variable for the limit in finding the derivative. The following incorrect and incomplete work was submitted on a homework assignment in which the problems were to find the derivative of $f'(x)$ when a. $f(x) = x^2 - 5$ and b. $f(x) = x^3 - 5$. Write a short discussion of all the student's error and suggest some ways for the student to avoid making these mistakes in the future.

a

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{((x^2-5)+(h^2-5))-(x^2-5)}{h}\\
&= \lim_{h\to 0}\frac{h^2-5}{h}\\
&= \lim_{h\to 0} h-\frac{5}{h}\\
&= ???.
\end{aligned}
$$

b

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{((x^3-5)+(\Delta x^3-5))-(x^3-5)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\Delta x^3-5}{\Delta x}\\
&= \lim_{\Delta x\to 0} x^2-\frac{5}{\Delta x}\\
&= ???.
\end{aligned}
$$

5. For each of the following functions express the derivative at 3 in functional notation, operator notation, and Leibniz notation. When possible find the derivative or give an estimate for the derivative.
   a. $y = f(x) = x^3 - 2x + 5$.
   b. $z = g(t) = \sin(t)$.
   c. $q = h(u) = 1/(u + 1)$.
   d. $y = f(x) = 2^x$.

6. Suppose $y = f(t)$ and $z = g(t)$. Express the following equations in function notation and operator notation.
   a. $y' = t^2 + 3t - 1$.
   b. $(\frac{dy}{dt})^2 = y+t^2$.
   c. $(\frac{dy}{dt})^2+(\frac{dz}{dt})^2 = 1$.
   d. $3\frac{dy}{dt} = z$; $\frac{dz}{dt} = -y$.

7. Find the following derivatives as indicated:
   a. $\frac{d}{dx}(3x^2-2x+5)$.
   b. $D_x(2x^2 + 5x)$.
   c. $(5x^2 - 2)'$.
   d. $\frac{d}{dx}(x^2-2x+5)\Big|_{x=0}$.

8. Other notations.
   a. The work of Newton and others interested in the physical motion interpretation of the derivative paid special attention to time as a controlling variable. Thus, when any variable, *x, y,* or *s* depends on time, *t*, in this notation the derivative is denoted by placing a dot over the variable. So for example we would write the derivative of *s* as $\dot{s}$. Find $\dot{s}$ for the following functions of time, *t*:
      i. $s=-16t^2 + 20t + 10$.
      ii. $s=6t^2 + 25t - 5$.
      iii. $s=3t^2 + 2t + 10$.

b. When many variables, say *t,x, y, u,* and *v,* control a single variable's value, say *z,* the controlled variable is described as a function of several variables. If we are able to keep all but one of these controlling variables constant while varying that one controlling variable, say *x,* the controlled variable can be considered a function only of *x.* In this case the derivative of *z* with respect to the variable *x* is often denoted by $\frac{\partial z}{\partial x}$. For each of the following variables, find the indicated derivatives.

i. $z=x^2y-xy+y$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

ii. $z=x^{-2}y-2^xy+17y$. Find $\frac{\partial z}{\partial y}$.