I.I.3 Extreme Values

Motivation: What weather report would fail to mention the highest and lowest temperatures of the day? What would baseball be without knowing who had the highest batting average for the season or who had struck out the largest number of times. And of course the pitchers with the most strike outs and the lowest earned run average are also worthy of mention. Or in golf the winner of a tournament has succeeded in taking the smallest number of strokes to complete the course. In track and field the question is who has run the fastest, jumped the highest, or thrown the farthest.

Even outside of sports, extremes are very important. In trading on the stock (or any other) market it is best to buy at the lowest price and sell at the highest price. When choosing consumer goods we look for the safest, the most energy efficient, the least expensive, and the most durable products. In judging entertainment the superlatives and prizes are always for " the best ....". Even in food and health product advertisements we hear claims of having the " highest content of ....." or the "lowest levels of ....." We live in a world of extremes and it is not likely to change.

Scientists and engineers of all kinds are also busy determining the extremes or limiting tolerances of materials and physical phenomena. They want to answer questions like what is the greatest speed for an object, the lowest temperature for a gas, the strongest material, the smallest detectible particle of matter. There seems to be no end to the list. The use of extremes can even lead to some very perplexing philosophical questions. Who is the least interesting person in the world? How can that person exist, since once designated as least interesting, that person is likely to be more interesting than the next most uninteresting person! Or in mathematics we can ask what is the smallest positive rational number or the largest real number with its square less than 2.

With all this interest in extremes, the mathematical question of extremes for the values of functions should seem reasonably important, and simple to state.

Definition: Given a function *f* with its domain D we say that **M is a maximum value for *f* if two conditions are satisfied**. First, **for all *x* in D, $f(x) \le M$**, i.e., M is what is called an **upper bound** for the function on the set D. The second condition is that there must be **some c^* in D where $f(c^*) = M$.**

We leave it to the reader to describe a comparable set of conditions to describe **$m=f(c_*)$** as a minimum value for *f*. See Figure *** for the mapping figure that visualizes the maximum and minimum value for *f*. The numbers where the function takes on the extreme values are call the extreme (maximum or minimum) points for the function.

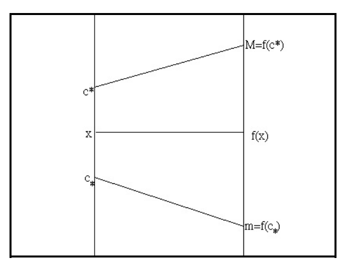

Figure 1

The extreme value questions for a function *f* are:

i) Do there exist real numbers c^* and c_* where $f(c_*) \le f(x) \le f(c^*)$ for all *x* in the domain of *f*?

ii) Can we find exactly or estimate any and all extreme points, i.e., those c's where f(c) is the maximum or a minimum value of f?

iii) Can we find exactly or estimate the maximum or minimim value of this function for its domain?

There are many situations where there is no maximum (or minimum) value for the function. For instance $f(x)= 1/x$ has neither a maximum nor a minimum value on its domain, and $f(x)=1/(1+x^2)$ has a maximum value of $1 = f(0)$ so 0 is a maxium point for f, but f has no minimum value on its domain even though its values are bounded below by 0. The exercises at the end of this section will explore further other situations where there are no extremes.

Fortunately the continuity of a function on a compact (closed and bounded) interval is enough to guarantee at least the existence of extreme values for a function. This result is stated below. An explanation of the reason for the result can be understood using either the graphical or the mapping figure interpretation of continuity. An informal proof of this result is included in Appendix ***.

The Extreme Value Theorem: Suppose f is a continuous function of a compact (closed and bounded) interval [a,b]. Then f has both a maximum and a minimum value for the interval, *i.e.*, there are numbers c^* and c_* [between a and b] were $f(c^*) \geq f(x)$ and $f(c_*) \leq f(x)$ for all x in [a,b].

With this theorem as a theoretical tool guaranteeing the existence of extremes for a continuous function on a compact interval, we can now try to find precisely what those values are and at what points they occur.

Looking at the graphs of some functions suggests that the tangent lines at the extreme points are horizontal, i.e., $f'(c) = 0$ when f(c) is an extreme value. This same conclusion makes sense for the mapping figure interpretation. It is very likely that at the extreme value a moving object would be changing the direction of motion, so the only reasonable instantaneous velocity at that moment, $f'(c)$, would be 0.

This would be the whole story if it weren't for the fact that a continuous function need not be differentiable at every point.

It is again the function $f(x) = |x-2| + 3$ which we can use to illustrate the difficulties. For the compact interval [-1,4] this continuous function has its minimum value of 3 at x=2 and its maximum value of 6 at x=-1 for the same

Compact Intervals: As we saw in the previous section, it is often useful to restrict the domain of numbers to a closed and bounded interval of the form $[a,b]$ where a and b are real numbers. These intervals are described technically by the term **compact** which brings to mind a sense of containment and boundedness. A compact interval is the only sensible way to consider the issue of intermediate values. A compact interval also seems appropriate for models that use time as a controlling variable. These models often refer to situations where the context is limited in both historical and predictive scope. We want to know what the extreme temperatures are for a day, a year, a decade, or over recorded history. We would like to predict the amount of rain for the coming week, month, or year. And for traveling, we want to know how far we have come since the start of our trip, and how much further we will go before we sleep. Not only do many questions arise for which compact intervals are the sensible domain for consideration, but also this type of interval has been adequate to insure important results like the intermediate value theorem and the extreme value theorem for continuous functions.

interval. Unfortunately this function is not differentiable at 2 and $f'(-1)=1$. So in some circumstances an extreme value may occur at c without $f'(c)$ being 0. The next theorem gives the result we need to proceed to find the extreme values for continuous functions on compact intervals.

Theorem I.I.4. (The Critical Point Theorem) Suppose that c is a point in the open interval (a,b) where $f(c)$ is an extreme value for f, then
either i) f is not differentiable at c
or ii) f is differentiable at c, and in this case $f'(c) = 0$.

Proof: If f is not differentiable at c then the first alternative is satisfied and the theorem is true. We may assume then that we are considering a situation where $f(c)$ is an extreme value for f and f is differentiable at c. Let's assume that $f(c)$ is actually the minimum value of f and leave the case when it is the maximum value as an exercise. Since $f(c)$ is the minimum value for $f(x)$, $f(x) \geq f(c)$ for any x, so $f(x) - f(c) \geq 0$ for any x. Consider $x > c$ and $x \to c$. See Figure ***.

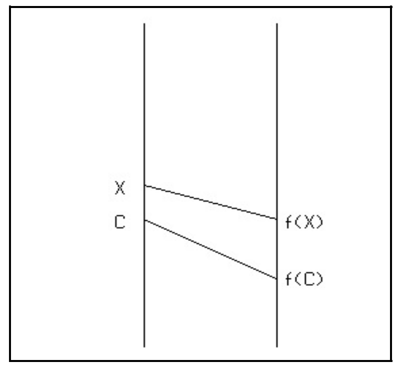

Figure 2

In this case $x - c > 0$ so $\frac{f(x)-f(c)}{x-c} \geq 0$. But this means that $f'(c)$ cannot be negative because there are numbers as close as we want to $f'(c)$ that are non negative. When we consider $x<c$ and $x \to c$ then $x - c<0$ so $\frac{f(x)-f(c)}{x-c} \leq 0$. But this means that $f'(c)$ cannot be positive. See Figure ***.
So the only possibility for $f'(c)$ is that $f'(c) = 0$, because we have assumed that $f'(c)$ exists.
EOP.

Definition: The two conditions that are in the conclusion of this theorem describe properties that characterize a point c as being "critical." We say c is a **critical point** for f if either of these two conditions hold, i.e., **if f is not differentiable at c or if $f'(c) = 0$.** If c is a critical point for f then we say that **$f(c)$ is a critical value for f**.

Comments: 1. With the definition of critical points and values we can express the critical point theorem briefly: Any extreme point for a function on an open interval is a critical point. Any extreme value for a function on an open interval is a critical value.
2. Don't confuse the critical point theorem with the statement that is its logical converse. It is certainly possible for a number to be a critical point without being an extreme point. As an example always remember that $x = 0$ is a critical number for $f(x)= x^3$, but certainly is not an extreme point for that function on any interval containing 0 as an interior point.

Application: Find the extreme values of f on [-2,3] where $f(x) = x^3 - 3x^2 + 2$.
Solution: Since f is differentiable, f is continuous and thus by the **extreme value theorem**, f has extreme values on the closed interval [-2,3]. Which points in the domain *could* give these

extreme values? If the extreme values occur for points in the open interval (-2,3) then the **critical point theorem** tells us they must occur at points c where f '(c) = 0. The only other possibility is that the extreme values occur at either -2 or 3, **the endpoints of the interval.** We find $f'(x) = 3x^2 - 6x$. So we solve

$$\mathbf{0 = f'(c) = 3c^2 - 6c = 3c(c - 2).}$$

Thus the only critical points are c=0 and c=2, both of which are in the interval [-2,3]. So the only points where an extreme value could occur are **-2, 0, 2 and 3**. But there is a maximum value and a minimum value guaranteed to exist already, so they must occur at one of the four points we've listed. Now we need only evaluate the function at the four points to find the maximum and minimum values as they will be the largest and smallest of these values.

$f(0) = \mathbf{2}$;
$f(-2) = (-2)^3 - 3(-2)^2 + 2 = -8 - 12 + 2 = \mathbf{-18}$;
$f(2) = (2)^3 - 3(2)^2 + 2 = 8 - 12 + 2 = -2$; and
$f(3) = (3)^3 - 3(3)^2 + 2 = 27 - 27 + 2 = \mathbf{2}$.

Upon examining these values we can conclude **the maximum value of f is 2**, while **the minimum value of f is -18.**

Method for Finding Extrema for Continuous Functions on Compact Intervals. Generalizing from the last example, we now articulate a procedure for finding the extrema of continuous functions when the domain is a compact interval.

Procedure for Finding Extrema for Continuous Functions on Compact Intervals.

Step 1. **Find all the critical points** for the function on the interval.
a. Find the derivative of the function.
b. List those critical points where the derivative does not exist.
c. List those critical points where the derivative is 0.

Step 2. Find **the value of the function at the critical points** and the **two endpoints** of the interval.

Step 3. The **largest value determined in step 2 is the maximum value** of the function on the interval. The **smallest value determined in step 2 is the minimum value** of the function on the interval.

This procedure gives a powerful tool for solving some problems that might arise in real situations. Despite the fact that word and story problems are sometimes a little far fetched and not too credible, they do provide some connection between the abstraction of functions defined on compact intervals and some contaxts where the calculus is applied in real practice.

Application. Old Mac Donald has a farm and on that farm there are some ducks. The ducks need a living area near the pond fenced on the land side to protect them from the larger animals.

The pond has a shore that is a straight line, so the farmer has decided to create a rectangular region for the duck pen using the pond as one border and 120 meters of fencing to enclose the other three sides. We will find the dimensions of the duck pen that will **enclose the greatest area.**

Solution: First **draw a figure** as in Figure 3 that illustrates the duck pen in a typical possible situation. The pond side is a line of undetermined length from the description of the problem, while we will use w to denote the measurement of the sides adjacent to the pond and l for the measurement of the length of the side opposite the pond. We let **A** denote the area of the rectangle as described so far.

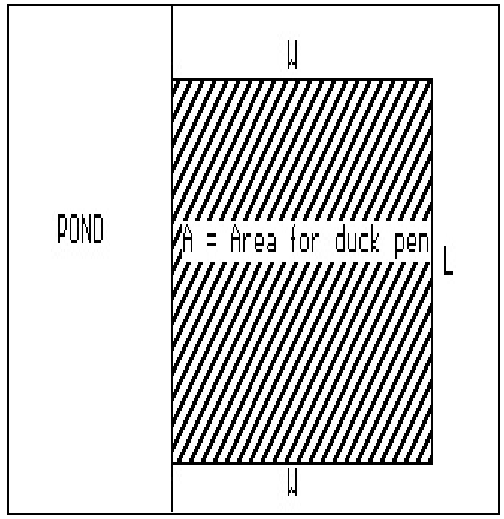

Figure 3

Our **problem** is to **find l and w so that A is as large as possible given the constraint that the length of fencing required, $l + 2w = 120$**, the total fencing allotted for the project. Certainly there is another constraint here, though not made explicit so far in the problem, namely both $l \geq 0$ **and** $w \geq 0$.

Finally it should be obvious that $A = l \cdot w$ is not an adequate way to express the area of the rectangle since it give the area as a function of two variables. So far the calculus we have developed is organized for functions of a single variable.

We can reduce the problem to a function of one variable by realizing that once we choose the value of w, the constraint equation tells us the value of l , $l = 120 - 2w$.

So $A = (120 - 2w)w = 120w - 2w^2$.

Now that we have the problem expressed in terms of a function of a single variable, we can consider whether the situation under discussion sets limits on the variables to allow application of the preceding procedure. Here again realizing that $l = 120 - 2w$ and that $l \geq 0$ we see that $60 \geq w \geq 0$, i.e. w is in $[0,60]$ and we can apply the procedure for finding extrema for continuous functions on closed intervals to this situation.

1. $A'(w) = 120 - 4w$. So the only critical point occurs when $w = 30$.
2. $A(0) = A(60) = 0$; $A(30) = (120 - 60)(30) = 1800$.
3. From 2 it is clear that the largest value of A is 1800 which occurs when $w = 30$ and hence $l = 120 - 2(30) = 60$.

Returning to complete the application, we have found that to achieve the largest possible area in the pen of 1800 square meters, the pen should have its sides beside the pen measured to be 30 meters while it side opposite the pond should be 60 meters.

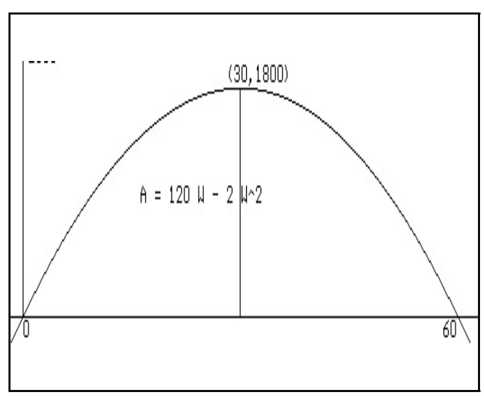

Figure 4

Comment: Although the last problem was solved using calculus to find the extreme value, the area function turned out to be quadratic. So the analysis could have been completed algebraically, without using calculus at all, by finding the coordinates of the vertex of the parabola that graphs the area function for the given interval. See Figure 4.

Application: The local soft drink company bottling department has just developed a new packaging material that will allow it to sell its product in cartons that are rectangular boxes. They want each container to hold precisely 1000 cubic centimeters (or 1 liter) of beverage and have determined for packing reasons that the top and bottoms will need to be double in thickness while the length of the top should be twice the width of the top. We will find the dimensions for the container that will use the least amount of material for its production.

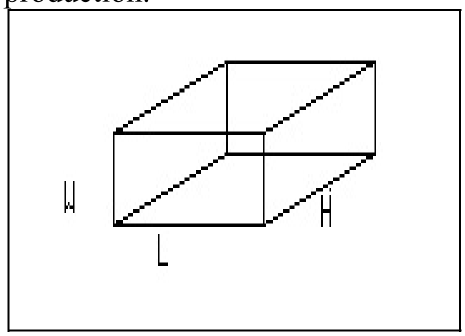

Figure 5

Solution: First draw a figure as in Figure 5 that illustrates a typical possible situation for the container. The height of the container is measured by **h** centimeters, the width of the top by ***w*** centimeters and the length of the top by *l* centimeters. We let A denote the amount of material needed to construct the container as specified in the situation.

Our **problem** is to **find h, *l*, and *w* so that A is as small as possible given the constraints that the container must hold a volume of exactly 1000 cubic centimeters, the top and bottom are double thickness and the length of the top is twice the width, i.e. $2w = l$.**

Certainly there are other constraints here, though not made explicit so far in the problem, namely that since the container must hold exactly 1000 cubic centimeters , so that $l \cdot w \cdot h = 1000$, it must also be that $l > 0$, $w > 0$, and $h > 0$.

It is only a beginning to express $A = 4\, l \cdot w + 2\, l \cdot h + 2\, w \cdot h$ since it gives the amount of material used as a function of three variables. We can reduce the problem to a function of one variable by realizing that once we choose the value of w, the constraint equations tell us the values of l and h, $l = 2w$ and $h = 1000/2w^2$.

So $A = 4\,(2w)\, w + 2\,(2w)\,(1000/2w^2) + 2\, w\,(1000/2w^2) = 8\, w^2 + 3000 / w$.

Now that we have the problem expressed in terms of a function of a single variable, we can consider whether the situation under discussion sets limits on the variables to allow application of the preceding procedure. Here the situation is more subtle than the previous application because there is **no obvious compact interval determined by the constraints** since we have only $w > 0$. To remedy this deficiency in the constraints will take a little thought about the situation and the continuity of A.

Certainly because A has an expression with w in the denominator of a fraction, as $w \to 0$, $A \to \infty$. This suggests that if we establish our own constraint for a lower bound on w, we won't have problems as long as this constraint is close enough to 0 to make A very large. Likewise a large value of w is not impossible from the constraints of the problem, but considering the fact that A has a term involving the square of w, a large value for w will have enormous impact on the size of A. This suggests we can establish our own constraint for an upper bound on w as long as this constraint is large enough to make A large. By these considerations of the constraints on w , we'll restrict our view of A as a function of w on the interval [.1,100].

Now we can apply the procedure for finding extrema for continuous functions on closed intervals to this situation.

1. $A'(w) = 16w - 3000/w^2$. So the only critical point in the given interval occurs when $w^3 = 375/2$. Let w^* denote $(375/2)^{1/3}$, which is approximately 5.72 , so the only critical point is w^*.

2. A(.1) = 30000.08 ; A(100) = 80030 ; $A(w^*) \approx 786.2$.

3. From 2 it is clear that the smallest value of A occurs when $w = w^*$ and is approximately 786.2 square centimeters.

Returning to complete the application, we have found that to use the smallest amount of material for the desired container we should have

$w \approx 5.72$ cm, $l = 2$, $w^* \approx 11.44$ cm and $h \approx 1000/\, 2w^{*2} \approx 15.72$ cm.

Application: Consider a random variable X which has a probability distribution function F(A) giving the probability that $X \le A$ for the interval $[-\pi/2 , \pi/2]$ by $F(A) = (1/2)[1 + \sin(A)]$. This is a continuous random variable on the interval since F is a continuous function with $F(-\frac{\pi}{2})=0$ and $F(\frac{\pi}{2})=1$. **A mode of a continuous random variable is the value(s) of A in the range of the random variable with the highest point probability density.**

A mode of a random variable is a number with the highest likelihood of the random variable falling close to that number. If we **let *f*(A) be the point probability density of X at A**, then $f(A) = F'(A)$ and in this example we have $f(A) = 1/2 \cos(A)$. To find the point with the highest point probability density in this example we can follow the procedure outlined in this section applied to f on the interval $[-\pi/2 , \pi/2]$. [See Figures 6 and 7.]

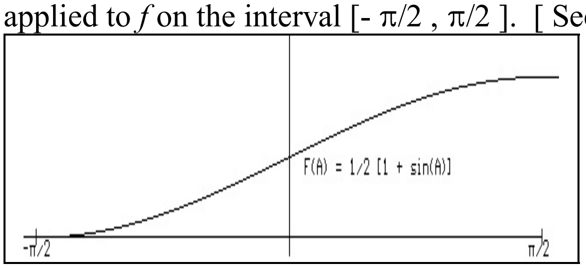

Figure 6

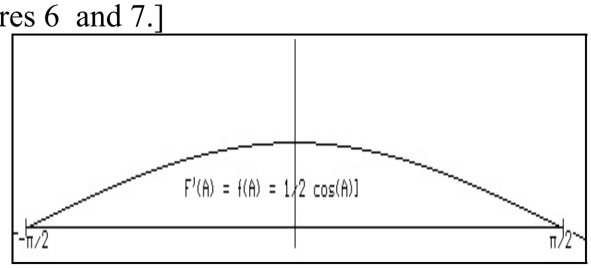

Figure 7

Step 1. $f'(x) = -1/2 \sin(x)$. So the only critical point in the interval is $x = 0$.

Step 2. $f(-\pi/2) = 0$; $f(\pi/2) = 0$ and $f(0) = 1/2$.

Step 3. Based on 2, the largest value for f occurs at 0. **So 0 is the mode of the random variable X .**

Exercises I.I.3

In exercises 1-8, for each of the functions find the extreme values on the indicated intervals.

1. $f(x) = x^2 - 4x + 3$; a) [-1, 5] b) [-1,1].
2. $f(x) = 2x^3 + 3x^2 + 4$; a) [-2,2] b) [1,3]

3. $g(t) = t^3 - 3t + 4$; a) [-2,2] b) [1,3]
4. $g(t) = \sin(t) + \cos(t)$; a) $[0, \pi]$ b) $[0, 2\pi]$.
5. $r(s) = s + 1/s$; [.1, 2].
6. $p(x) = x^4 - 2x^2 + 2$; [-2,2].
7. $f(x) = x - \ln(x)$; [1/2, 2].
8. $g(x) = x^2 - 2e^x$ [-1,1]

9. A little pig was going to build a two room house of wood. The floor plan was a rectangle with one partitioning wall containing 100 square feet. The pig found that the cost for the outside wall was \$3 per linear foot while the inside wall cost only \$2 per linear foot. Find the dimensions of the house that would minimize the total cost for the walls.

10. A little pig was going to build a two room house of brick. The floor plan was a rectangle with one partitioning wall of wood containing 100 square feet. The pig found that the cost for the outside wall was \$5 per linear foot while the inside wall cost only \$2 per linear foot. Find the dimensions of the house that would minimize the total cost for the walls.

11. Find the dimensions of the rectangle of largest area that can be inscribed in a right triangle with sides of length 3,4,and 5.

12. A ball is thrown in the air so that its height above ground level is $f(t)$ feet at time t seconds where $f(t) = -16t^2 + 64t$. Find the maximum height of the ball.

13. Use calculus to show that the vertex of the parabola with equation $Y= A X^2 + B X + C$ occurs when $X = -B/2A$.

14. More Pigs: One little pig decided to build his house with the scrap wood he found in the lumber yard. He wants to match the pigs in problems 7 and 8 by having two rooms in a rectangular house. He can only find enough wood to set up walls and a partition with a total of linear footage of 80 feet. Find the dimensions of the house that will give this pig the largest floor area in the house.
15. The pig in problem 7 has changed his mind. Because of financial considerations he has budgeted \$400 for his costs of materials for the walls of his house. He still wants to build a two room rectangular house with wood. Find the dimensions of the house that will have the largest floor area under these constraints.
16. The pig in problem 8 has become more cost conscious and has decided to budget his costs for materials (bricks and wood) for the walls of his house at \$800. He still wants to build a two room rectangular house. What are the dimensions of the house that will have the largest floor area under these constraints ?

17. A rectangle is inscribed in the parabola determined by the equation Y = 9 - X^2. Its base is placed on the X-axis as in Figure 8. Find the dimensions of the rectangle that will enclose the region of largest area for this situation.

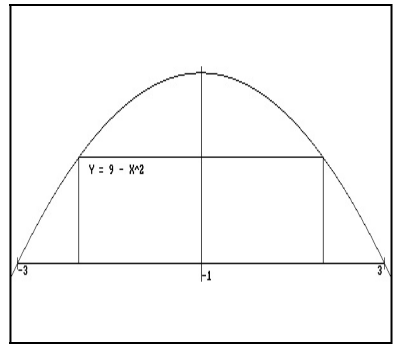

Figure 8

18. (Three room house project) The pigs have all changed their minds and now want to build three room rectangular houses with rectangular rooms. Discuss the possible floor plans for their new projects.
    a. Find the dimensions for the projects with the 100 square foot constraint of problems 7 and 8 minimizing costs.
    b. Find the dimensions for the projects with the materials and cost constraints of problems 12, 13, and 14 maximizing floor area.
19. Each of the following functions fails to have an extreme value for the interval [0,2] Discuss briefly why this failure happens.

a. $f(x) = \begin{cases} 1 & \text{if } x=0 \\ 2x & \text{if } 0<x<2 \\ 1 & \text{if } x=2 \end{cases}$ c. $f(x) = \begin{cases} 1 & \text{if } x=0 \\ \dfrac{1}{2x-x^2} & \text{if } 0<x<2 \\ 1 & \text{if } x=2 \end{cases}$

b. $f(x) = \begin{cases} 2x-x^2 & \text{if } x\neq 2 \\ \dfrac{1}{2} & \text{if } x=2 \end{cases}$

Appendix *: Informal Explanations [Proof] of the Extreme Value Theorem**

Assume *f* is continuous on [a,b]. The condition of continuity on a compact interval guarantees the function is bounded above and below. From the point of view of the graph this is true because the graph can be thought of as a curve that is drawn without lifting a pencil from the paper from (a,*f*(a)) to (b,*f*(b)). Given the ability to scale the drawing appropriately, we can fit the drawing inside a picture frame or a window. This means we have bounded the values of $f(x)$ by some numbers B and S so that for any x in the domain, S < $f(x)$ < B. See Figure 9.

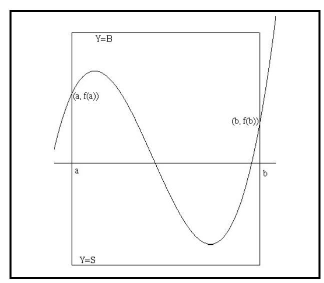

Figure 9

For the mapping figure interpretation the continuity means we can interpret *f*(t) as the position of a physical object moving on a line from point *f*(a) to point *f*(b). This allows us to take a large enough scale for the target line to view the object during the motion at all times. Again we have found numbers B and S where S < $f(x)$ < B for all x in the domain. See Figure 10.

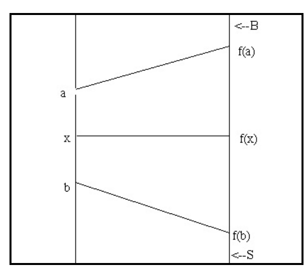

Figure 10

Now that we have bounds for $f(x)$, we can find the maximum value by taking numbers successively that are midway between the last upper bound and a number that is not an upper bound. Each time we find either a smaller upper bound or a larger number that is not an upper bound. This process will lead to a single real number M which must be an upper bound and for which any smaller number is not an upper bound for the values $f(x)$. [We have found the least upper bound of the values $f(x)$.] A similar argument will lead to a number L which is the greatest lower bound for the values $f(x)$.

The last part of the argument is to show that there is some c^* and c_* where $f(c^*) = M$ and $f(c_*) = L$. Here the graphical interpretation suggests this is correct because if the line Y = M does not meet the curve that is the graph of $f(x)$ then there would be some smaller number which would still be an upper bound for the values $f(x)$ and M was supposed to be the least upper bound. So the curve does meet the line Y = M at least once, and at that point we find c^* where $f(c^*) = M$.

An argument based on the mapping figure interpretation is similar. If the moving object does not ever reach the point on the target with coordinate M, then there must be some point on the target that is an upper bound for $f(x)$ but is lower than M which contradicts the fact that M is the least upper bound. Thus at some time c^*, $f(c^*) = M$.

Similar arguments justify the existence of c_* where $f(c_*) = L$.

Thus the continuous function f has extreme values for [a,b]. **EOP**