I.C.2 Economics and Marginal Analysis.

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Our initial discussion of mathematical models in Section 0.C. introduced a model for investigating change in an economic context. The key tool for this analysis was **the margin** in cost, revenues, or profit **for an additional unit of production**. In this section we continue exploring marginal analysis when relatively small changes are made in production, the controlling variable.[1]

x	C(x)	MC(x)
100	300	3
101	303	-
1000	2500	2
1001	2502	-

Recall from Section 0.C, that we had discussed The Umbrella Company (TUC), a small company that manufactures umbrellas with artistic prints on the fabric. Focusing on TUC's production costs and profits as functions of the number of umbrellas produced, we considered average costs and profits and average marginal costs and profits. [At this point rereading the examples in Chapter 0.C might be helpful.]

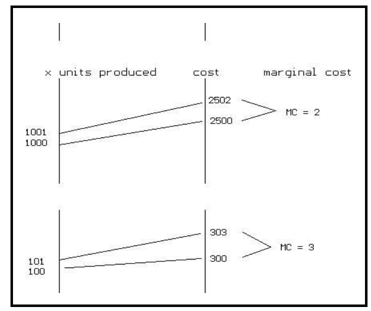

Figure 1

Marginal cost. Let's focus our discussion on *marginal costs*, recognizing that there are many other marginal concepts in economics.[2]

When the units of production are indivisible, like an umbrella, **the marginal cost is the added cost (per unit) for producing one more unit-** one more umbrella. We can represent the concept of marginal costs in several ways. Let's suppose that the total cost of producing x umbrellas is C(x) dollars and the marginal cost of producing one more umbrella when you have produced x umbrellas is MC(x) dollars (per unit), so

$$MC(x)=C(x+1) - C(x).$$

So if C(100) = 300 and C(101) = 303, then MC(100) = 3,
while if C(1000) = 2500 and C(1001)= 2502, then MC(1000)= 2.

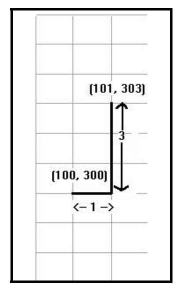

Figure 2

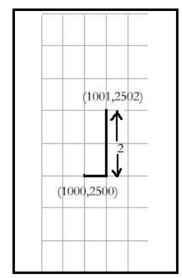

Figure 3

In Table 1 and Figures 1-3 we see how this information is represented with a table, mapping figures, and graphs. No matter how we represent the information, **the important thing to notice is that the marginal cost varies depending on the production level.**

Average marginal cost. In the description of the TUC story, we let Δx denote the number of additional umbrellas produced and ΔC denote the increase in total cost

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[1] mar·gin·al (mär"j_-n_l) adj. 1. Of, relating to, located at, or constituting a margin, a border, or an edge: the marginal strip of beach; a marginal issue that had no bearing on the election results. 2. Being adjacent geographically: states marginal to Canada. 3. Written or printed in the margin of a book: marginal notes. 4. Barely within a lower standard or limit of quality: marginal writing ability; eked out a marginal existence.
Economics 5. a . Having to do with enterprises that produce goods or are capable of producing goods at a rate that barely covers production costs. b. Relating to commodities thus manufactured and sold.
Psychology 6. Relating to or located at the fringe of consciousness.
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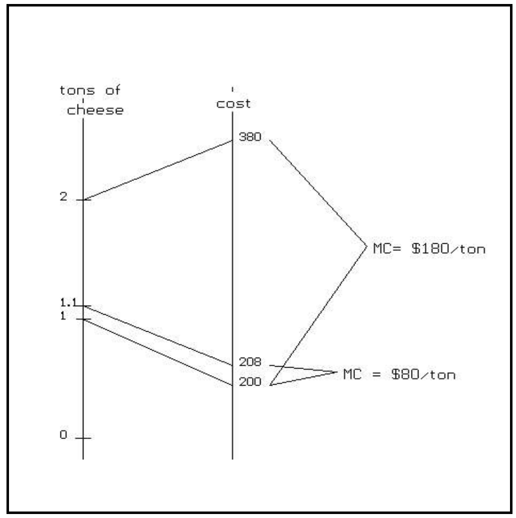
[2] For instance there are marginal revenues and profits generated by sales as well as marginal productivity of labor, marginal propensity to consume and spend, marginal propensity to save, and marginal utility. You might investigate these further in any beginning college economics textbook.

resulting from the increased production. The **average marginal cost,** $\overline{MC}$**,** was defined in Section 0.C as the ratio of the change in cost, ΔC, compared to the number of additional units produced Δx, so $\overline{MC} = \frac{\Delta C}{\Delta x}$. See Figure ***. It is an average of the individual marginal costs in a change in production of more than a single unit.

From your own experiences at making things it should seem sensible that the marginal cost of producing umbrellas might vary with the level of production. For example if we can buy designer fabric more cheaply in larger quantities, the cost of an additional umbrella when we are producing 1000 umbrellas might be noticeably less than the cost of an additional umbrella when we are producing only 25 umbrellas.

Units of production. In the umbrella example a single umbrella is the ***indivisible*** unit of production. Umbrellas make sense only as whole units; that is, we would not consider producing 1/2 or 5/6 of an umbrella. For some commodities, such as milk or cheese, the units of production are continuously (perfectly) divisible measurements such as gallons, liters, pounds, kilograms, or tons. We can certainly make sense of a 1/2 gallon of milk and 3.56 tons of cheese. In the case where the unit of production is (perfectly) divisible, a more careful analysis determines the marginal cost of production.



Redo ! **Figure 4**

Motivation: The average marginal cost can depend on the choice of units. How does this happen? Suppose, for example, that the Fresh Cheese Company produces 3 tons of cheese for a cost of $2000, 3.1 ton of cheese for $2075, and 4 tons of cheese for $2600. Let's consider the marginal cost at the 3 ton level of production with differing choices for the unit. Using one ton as the unit of production, we look at the difference in cost from 3 to 4 tons, $600, as the cost of the additional unit. So the marginal cost **per ton** is $600 **per ton**. Changing the unit of production to 0.1 ton (200 pounds), we now consider the difference in cost from 3 to 3.1 tons, $75, as the cost of the additional unit. Thus the marginal cost **per 0.1 ton** is $75 **per 0.1 ton**. The marginal cost based on a unit of 0.1 ton when treated as a ratio is equivalent to a rate of $750 per ton. So at the same level of production (3 tons) we have found different marginal costs per ton depending on the choice for the unit. See Figure ***.

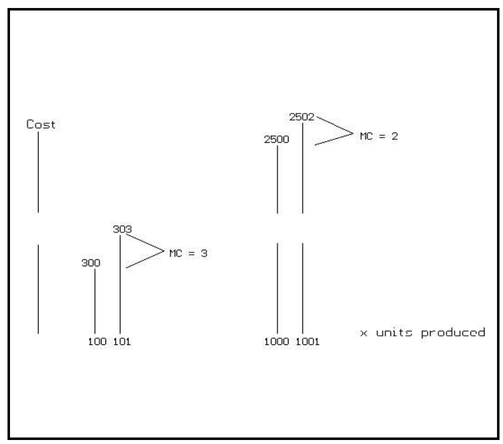


Figure 5

Which of these rates would be more useful for estimating the additional costs for some small change in cheese production? It should seem sensible to use the rate that is based on the smaller unit of change, even when the rate ($750 per ton) is stated in terms using the more standard unit of dollars per ton.

Peter Pan: The sequel! To give a more accurate estimate of additional costs, we could use an even smaller unit for the computations. So we can see that for continuously measurable goods (such as cheese, not umbrellas) the marginal cost is not determined by a single intrinsic[3] unit. That is - umbrellas have an intrinsic unit for measurement- **one umbrella,** but cheese does not have a single intrinsic unit for measurement- one pound, one kilogram, and one ton are equally sensible units depending on the context.

In fact, for goods that can be measured continuously, many economists define the **marginal cost** as **the rate of change** (in cost per unit production) **estimated by average marginal costs based on arbitrarily small increments of production.** So we have come again to a process of estimation, this time using smaller and smaller units to measure the corresponding increments in production costs. To maintain a common unit for reporting the production level and the marginal cost, we rescale the ratios to a common unit.

Here then is a summary for estimating the marginal cost of production at a level of *a* tons of cheese where C(x) is the cost of producing x tons:

Let h tons be the unit for consideration where h is small. [$h \approx 0$ or $h \to 0$.] The change in cost for one additional unit of production is $C(a+h) - C(a) = \Delta C_a(h)$ which is the unit marginal cost for this single unit. To rescale this to the **standard** one-ton unit, we divide by h to find the *standardized* unit marginal cost,

$$\overline{MC_a}(h) = \frac{\Delta C_a(h)}{h} = \frac{C(a+h)-C(a)}{h}.$$

Notice that $\overline{MC_a}(h)$ also represents the *average* marginal cost for a production change of h tons.

When h is close to 0, the definition of marginal cost means that $\overline{MC_a}(h)$ will be close to the marginal cost of production, $MC(a)$. Symbolically, when $h \to 0$, $\overline{MC_a}(h) \to MC(a)$.

Note: If we introduce the variable x by letting $x = a + h$ then $h = x - a$. When we consider $h \to 0$, then we have $x \to a$ and the marginal cost for h additional tons can be approximated by

$$\frac{C(x)-C(a)}{x-a}..$$

This expression has a form that we have encountered in other contexts.

ExampleI.C.1. Marginal Cost: The Costly Food Corporation (CFC) has been reviewing the costs for producing its most popular breakfast cereal, Costly Corn Flakes . The research department has come up with a production cost model that uses a quadratic function. With this model, the estimated cost of producing *x units* of Costly Corn Flakes is C(x) dollars where $C(x)=x^2-5x+80$ Each unit is 100 pounds.

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[3]intrinsic: adjective belonging to the basic nature of someone or something; essential.

CFC is currently producing 3000 pounds (30 units) of Costly Corn Flakes in a week. We will find (i) the **average marginal cost per pound** for producing one unit (100 pounds) as well as (ii) the **marginal cost per pound** based on this model.

(i) To find the average marginal costs we use the cost function suggested by the naked model and the formula $\overline{MC}_a(h) = \frac{\Delta C_a(h)}{h} = \frac{C(a+h)-C(a)}{h}$ with a=30, h=1 or h=-1. Here are the calculations:

$$C(30)=(30)^2-5\cdot 30+80=900-150+80=830$$
$$C(31)=(31)^2-5\cdot 31+80=961-155+80=886$$
$$C(29)=(29)^2-5\cdot 29+80=841-145+80=776$$

So, $\overline{MC}_{30}(1) = \frac{\Delta C_{30}(1)}{1} = \frac{C(30+1)-C(30)}{1} = 886-830 = 56$ and

$\overline{MC}_{30}(-1) = \frac{\Delta C_{30}(-1)}{-1} = \frac{C(30-1)-C(30)}{-1} = \frac{776-830}{-1} = 54.$

(ii) For the marginal cost we start with a little algebra to find the average marginal cost when a = 30 and h is variable, *i.e.*, we will find $\overline{MC}_{30}(h) = \frac{\Delta C_{30}(h)}{h} = \frac{C(30+h) - C(30)}{h}$

$$\begin{aligned} C(30 + h) &= (30 + h)^2 - 5\cdot(30 + h) + 80 \\ &= 900 + 60h + h^2 - 150 - 5h + 80 \\ &= 830 + 55h + h^2 \end{aligned}$$

So,

$$\begin{aligned} \Delta C_{30}(h) &= C(30 + h) - C(30) \\ &= (830 + 55h + h^2) - 830 \\ &= 55h + h^2 \end{aligned}$$

and thus,

$$\overline{MC}_{30}(h) = \frac{\Delta C_{30}(h)}{h} = \frac{55h + h^2}{h} = 55 + h.$$

Note: The work we did above in (i) with $h = \pm 1$ is consistent with this cool formula.

To complete finding the marginal cost at $a = 30$ we consider the average marginal cost when h is close to 0. That is, when h is close to 0, $\overline{MC}_{30}(h)$ will be close to the marginal cost of production, $MC(30)$. Symbolically, when $h \to 0$, $\overline{MC}_{30}(h) = 55 + h \to 55 = MC(30)$.

Now it should make sense from the algebraic simplification of $\overline{MC}_{30}(h)$ that the marginal cost at a = 30, MC(30), is $55 per 100 pounds.

ExampleI.C.2. Marginal Revenue: Costly Food has also been reviewing the revenues generated by sales of its most popular lunch snack, Costly California Wraps (CalWraps™). Once again their famous research department has come up with a naked sales model that uses a linear demand function where *x* is the weight (measured in pounds) of CalWraps expected to be sold per week at a price of *p* dollars per pound and $p = 10 - 0.002x$. With this model, the estimated revenue from sales of *x* pounds at a price of *p* dollars is $R(x)$ dollars where

$$R(x) = x \cdot p = x \cdot (10 - 0.002x) = 10x - 0.002x^2.$$

CFC is currently selling 3000 pounds of CalWraps in a week at a price of $4.00 per pound. We will find (i) the **average marginal revenue per pound** as well as (ii) the **marginal revenue per pound** based on this model.

(i) To find the average marginal revenue we use the revenue function suggested by the naked model and the formula $\overline{MR_a}(h) = \frac{\Delta R_a(h)}{h} = \frac{R(a+h) - R(a)}{h}$ with a=3000, h=1 or h= -1. Here are the calculations:

$$R(3000) = 10(3000) - 0.002(3000)^2 = 12{,}000$$
$$R(3001) = 10(3001) - 0.002(3001)^2 = 11{,}997.998$$
$$R(2999) = 10(2999) - 0.002(2999)^2 = 12{,}001.998$$

So, $\overline{MR_{3000}}(1) = \frac{\Delta R_{3000}(1)}{1} = \frac{R(3000+1) - R(3000)}{1} = 11\,997.998 - 12{,}000 = -2.002$

and

$$\overline{MR_{3000}}(-1) = \frac{\Delta R_{3000}(-1)}{-1} = \frac{R(3000-1) - R(3000)}{-1} = -(12\,001.998 - 12{,}000) = -1.998.$$

Note that according to the research department's model for demand, to sell 3001 pounds the price will be 10 - 0.002 * 3001 = $3.998 per pound. While to sell 3000 pounds the price will be $4.00 per pound.

(ii) For the marginal revenue we start with a little algebra to find the average marginal revenue when a = 3000 and h is variable, *i.e.*, we will find

$$\overline{MR_{3000}}(h) = \frac{\Delta R_{3000}(h)}{h} = \frac{R(3000+h) - R(3000)}{h}$$

$$\begin{aligned} R(3000+h) &= 10(3000+h) - 0.002(3000+h)^2 \text{ [4]} \\ &= (3000+h)(10 - 0.002(3000+h)) \\ &= (3000+h)(4 - 0.002h) \\ &= 12000 - 2h - 0.002h^2 \end{aligned}$$

So,

$$\begin{aligned} \Delta R_{3000}(h) &= R(3000+h) - R(3000) \\ &= (12000 - 2h - 0.002h^2) - 12000 \\ &= -2h - 0.002h^2 \end{aligned}$$

and thus,

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[4] We used the common factor of (3000 + *h*) in this to avoid squaring (3000 + *h*).

$$\overline{MR_{3000}}(h) = \frac{\Delta R_{3000}(h)}{h} = \frac{-2h - 0.002\,h^2}{h} = -2 - 0.002\,h.$$

Note: The work we did above in (i) with $h = \pm 1$ is consistent with this hot formula.

To complete finding the marginal revenue at $a = 3000$ we consider the average marginal revenue when h is close to 0. That is, when h is close to 0, $\overline{MR_{3000}}(h)$ will be close to the marginal revenue for sales, $MR(3000)$.

Symbolically, when $h \to 0$, we see that $\overline{MR_{30}}(h) = -2 - 0.002\,h \to -2$. So $MR(3000) = -2$.

Now it should make sense from the algebraic simplification of $\overline{MR_{3000}}(h)$ that the marginal revenue at $a = 3000$, $MR(3000)$, is -\$2 per pound. That is, for an additional sale of one pound of CalWraps the revenue decreases \$2 because of the lowered price needed to make that additional sale as we saw in the work for part (i).

Exercises I.C.2. [**Work in Progress.- Problems need to be added here for estimating marginals comparable to examples and exercises in I.C.1.**]

Problem??? If the individual marginal costs for each additional unit of production are accumulated, we will have determined the net change in costs, ΔC. If then ΔC is divided by the number of additional units produced, Δx, we have found the average of the individual marginal costs.

Writing Project: Write a comparison between average velocity and average marginal cost. For the total distance traveled (total cost) we can find an average velocity (average cost), while for a short additional part of a trip (additional production) we might find the average velocity for that addition only (as we find the average marginal cost for that additional production).

In your comparison use graphs and mapping figures to illustrate your remarks.

Project on Estimation and Archimedes: The Greek mathematician and scientist Archimedes (287-212 B.C.E.) wrote much using a systematic method for estimation. Read *On the Measurement of the Circle* where Archimedes estimates π in Proposition 3. Write a paper describing the content and method of this work. [Move to end of I.C ?]