Chapter 0.B.3. [More than Just] Lines.

Of course you've studied lines before, so why repeat it one more time? Haven't you seen this stuff about lines enough to skip this section? **NO!** But why? It is true that after you catch on to how lines and constant rates of change work, this stuff actually makes some sense and is not too hard. And perhaps because of its ease and sensibility, lines and linear functions have become a foundation of much of mathematics, especially the part described by the calculus.

In a sense, calculus can be thought of as an attempt to study non-linear functions and curves and non-constant rates by approximating them with a simpler linear mathematics. Our purpose in this section is to review some of this simpler mathematics, to explore some of the tools for functions established in the previous section, and to set the stage for the calculus by investigating the tangent problem with algebra for some simple examples. We will pursue the tangent problem more thoroughly in Chapter I using the analysis of the calculus.

The algebra of a line in the plane: In 1637 Descartes' introduction of algebra into the analysis of geometry redirected the path of mathematical investigations since then. The main thrust of this work was that algebraic relationships between measurements could be built based on the use of reference lines and established units of measurement. Let's apply this approach to a line in a coordinate plane to (re)discover a (linear) equation relating the coordinates (a,b) of any point on that line. For this analysis consider two cases: (i) when the line is parallel to a coordinate axis and (ii) when the line in not parallel to either coordinate axis.

To begin, let's suppose we can determine one point on the line, say (5,3) and that the line is parallel to the Y-axis. Then the first coordinate of any point (a,b) on the line will be the same, that is, 5, while the second coordinate is quite arbitrary. The equation then that describes a general point on this line has the form a =5. Thinking of this a as a variable we express the equation with the general form X=5.

If instead we suppose the line is parallel to the X axis then by a similar argument it should make sense that the equation that corresponds to this line is Y= 3. See Figure 1.

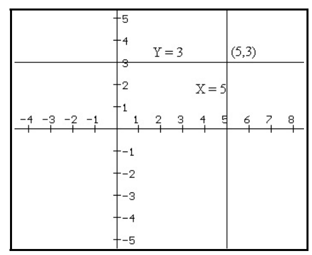

Figure 1

Now we'll consider the case where the line is not parallel to either axis. Suppose the line passes through (2,3) and (5,1). Using lines parallel to the axes we can form similar triangles with the distinct point (a,b) on the line and form ratios from the lengths of corresponding sides, so that $\frac{3-1}{3-b} = \frac{2-5}{2-a}$.

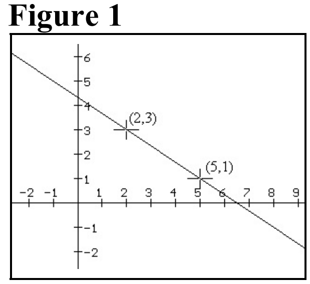

Figure 2

This leads directly to the equation $2(2-a)+3(3-b)=0$ or 13 = 2a + 3b. Treating a and b as variables and replacing them symbolically with X and Y we have 2 X + 3Y = 13.

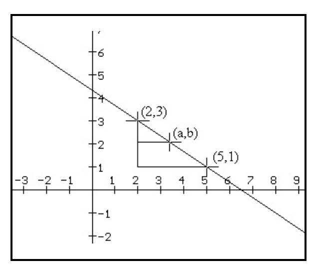

Figure 3

Following the method we've illustrated above shows that knowing the coordinates of two points on any line allows you to find an equation of the line that characterizes the coordinates for all points that lie on the line. Thus lines will correspond to linear equations of the form AX + BY = C and it can be shown conversely that the graph of any nonzero linear equation will correspond to a line in the coordinate plane.

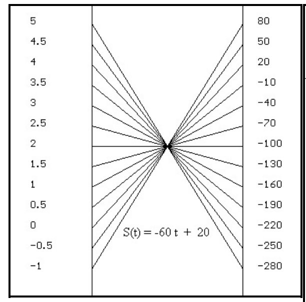

Figure 7

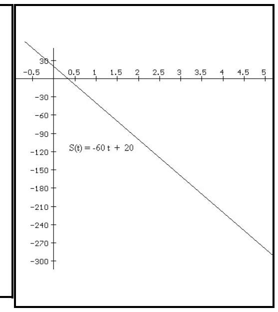

Figure 8

The General Linear Equation: You have probably been asked the question, "which came first, the chicken or the egg?" Of course there is no way to determine such a priority, and for many pairs of variables there is also no way to determine which should be considered the controlling variable.

Take for example the motion of the minute hand of a clock. Is it the time that determines the position of the hand or is it the hand that allows us to determine the time? Or consider the common example of changing the measurement of temperatures between Fahrenheit and Centigrade scales. Is there any reason to give priority to the Fahrenheit measure except for the cultural bias of our experience? And what about the list of other scale changes such as between miles and kilometers, gallons and liters, or even dollars and yen. All of these variable relations can be expressed without bias using the general linear equation: $AX + BY = C$ where A, B, and C are constants not all zero while X and Y are the variables.

We can determine the appropriate constants for such a relation by using any two pairs of data for which the equation is true. Let's follow this through for temperature. We use the variables F for Fahrenheit and G for Centigrade so the general equation can be written as $AF + BG = C$. We note for the data that the freezing temperature of water corresponds to F=32 and $G = 0$ while the boiling point is measured by F =212 and $G = 100$.

This gives the pair of equations A 32 + B 0 =C and A 212 + B 100 = C. Now using these equations we can see that A = 1/32 C and $B=(\frac{1}{100} - \frac{212}{3200})C$ so the relation can be expressed as $\frac{1}{32}CF + (\frac{1}{100} - \frac{212}{3200})CG = C$. Simplifying this last equation by factoring and cancelling C we arrive at $\frac{1}{32}F - \frac{180}{3200}G = 1$, so $100F - 180G = 3200$ or $5F - 9G = 180$.

With this equation we can rule the world! - or at least convert between these two scales without difficulty. Figure 4 visualizes this relation with a mapping figure showing the correspondence at some selected values.

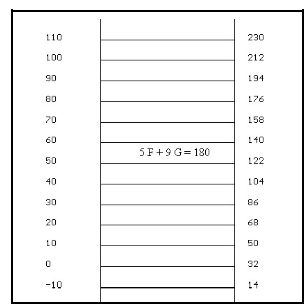

Figure 4

Linear Functions and Constant Rates: At 3:00 *p.m.* a car is 20 miles north of San Francisco travelling north on U.S. 101 at a constant rate of 60 miles per hour. The position of the car after t hours can be described as a linear function of time, $S(t)=60\,t + 20$. Of course this function is not correct for all values of t, but is accurate as long as the car continues to travel at 60 mph. If the car were heading south at 60 miles per hour the function would still be linear but in this case the value would be $S(t)= - 60\,t + 20$. A negative number for the function indicates the car is south of San Francisco. Here are

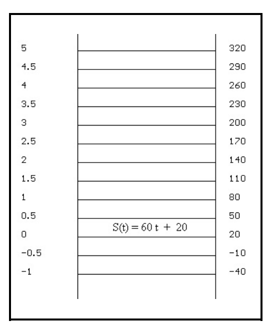

Figure 5

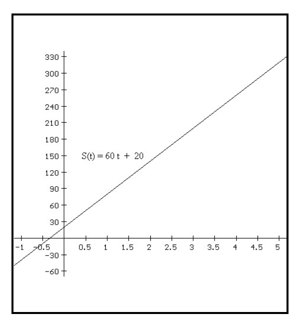

Figure 6

mapping figures and graphs to visualize these two linear functions. See Figures 5-8. Notice that these linear functions could also be used to describe the cost of producing x items at 60 dollars per item with an initial cost of 20 dollars or the level of water above the adequate supply line in a water tank in centimeters that starts at 20 centimeters above the level and is decreasing at a rate of 60 centimeters per hour.

The rates in each of these interpretations are interpreted geometrically as the slope of the line, m, which give the ratio of the change in the second coordinates, Δy, of pairs of points on the line to the corresponding change in the first coordinates, Δx, of those points. In geometry the slope is described sometimes as the ratio of rise to run, or $m = \frac{rise}{run} = \frac{\Delta y}{\Delta x}$.

It is often useful to determine an equation expressing a linear function based on the rate or slope m together with a single point that corresponds to the function's evaluation, (c,d). The function can be described by thinking in terms of the position of an object moving on a straight line. At time $t = c$, the object is at position $S(c) = d$ traveling at a rate of m. So the function satisfies the equation $S(t) = d + m(t - c)$. Thus a linear function with slope 6 passing through the point (3,9) can be expressed by the equation $S(t) = 9 + 6(t - 3) = 6t - 9$.

Solving a tangent problem using geometric and algebraic methods.

One of the problems that was crucial in the development of the calculus was the tangent problem, *i.e.*, finding a general method for describing a line tangent to a given curve at a given point on the curve. With the development of coordinate geometry following Descartes' program for using algebra to study geometry, interest in finding a solution to this problem increased dramatically. The focus of the problem changed to finding the **slope** of a line **tangent** to a given curve at a specific point on that curve.

Temporarily we'll use a geometric characterization that a tangent line will meet a curve only at one point. **[We will have to abandon these approaches in Chapter I, so don't become too committed to these examples for their methods.]** For now we will examine only circles and parabolas, curves for which the solution of the tangent problem was know to the Euclid and Archimedes. The algebraic method we use might be pursued further, but it is not as powerful as the methods that are found using the calculus, so we introduce it here only to provide some background experience with the tangent problem and at least two families of curves for which the problem is solved, circles and parabolas.

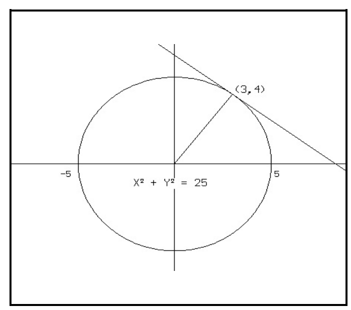

Figure 9. $\mathbf{X^2 + Y^2 = 25}$

Example (The Circle): Find the slope of the line tangent to the graph of $\mathbf{X^2+Y^2 = 25}$ at the point (3,4). [See Figure 9.]

Solution (Geometric): The graph here is a circle of radius 5 with its center at the origin. From Euclidean geometry we know that the tangent line is perpendicular to the radius line from the center (0,0) to the point (3,4). Since the radius line has a slope of 4/3, we conclude that the slope of this tangent line is -3/4 [so that the product of these slopes is -1 as it should be for perpendicular lines].

Example (A Parabola): Find the slope of the line tangent to the graph of $\mathbf{Y = X^2}$ at the point (3,9). [See Figure 10.]

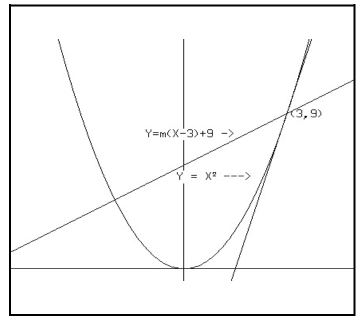

Figure 10. $\mathbf{Y = X^2}$

Solution (Algebraic): Suppose m is the slope of a line passing through the point (3,9). Then the equation of the line is given by

$$\mathbf{Y = 9 + m (X - 3). \ (*)}$$

When this line intersects the curve $Y=X^2$ we see by substituting X^2 for Y and subtracting 9 from both sides of the equation (*) that

$\mathbf{X^2 - 9 = m (X - 3). \ (**)}$

By factoring this last expression we find that

$\mathbf{(X+3)(X-3) = m (X - 3) \ \ (***)}$

or with a little more algebra we come to the equation

$$\mathbf{(X + 3 - m)(X - 3) = 0. \ \ (****)}$$

Thus either $X + 3 - m = 0$ or $X - 3 = 0$. But for the tangent line there should be only one point of intersection of the curve with the line, namely when $X = 3$. Thus we must have that the solution of the equation $X + 3 - m = 0$ for X must be $X = 3$. So we let $X = 3$ in the equation and we see that in that case $m = 6$.

Notice that $X + 3 = m$ for all choices of $X \neq 3$ This formula gives a clear correspondence between values of X and m, so that for each $m \neq 6$ we can find an $X = m - 3$ which will be different from $X = 3$, and which will be the **first coordinate of a second point** lying on both the line and the graph of $Y = X^2$. For example, when $m = 5.5$, the line will pass through a second point on $Y = X^2$ with first coordinate 2.5.

Note finally that using $m = 6$ makes the equation of the tangent line $Y= 9 + 6(X-3)$, and it is not hard to check that this line does intersect the graph of $Y = X^2$ only at the point (3,9).

Example (A parabola at any point): Now find the slope and an equation of the line tangent to the graph of $Y = X^2$ at the point (a,a^2).

Solution: Following the outline above, we replace each "3" with an "a" and each "9" with an "a^2". Suppose m is the slope of any line passing through the point (a,a^2). Then the equation of the line is given by $\mathbf{Y = a^2 + m (X - a). \ \ (\#)}$

When this line intersects the curve $Y = X^2$ we see by substituting X^2 for Y in **(#)** that

$$\mathbf{X^2 - a^2 = m (X - a). \ \ (\#\#)}$$

By factoring this last expression we find that

$\mathbf{(X + a) (X - a) = m (X - a) \ \ (\#\#\#)}$

or with a little more algebra we come to the equation

$\mathbf{(X + a - m) (X - a) = 0. \ \ \ (\#\#\#\#)}$

Thus either $X + a - m = 0$ or $X - a = 0$. But for the tangent line we want only one point of intersection, namely when $X = a$, so

$\mathbf{a + a - m = 0 \text{ or } m = 2a.}$

As before, an equation of the tangent line is $\mathbf{Y = a^2 + 2a (X - a)}$ and you can verify that this line intersects the graph of $Y = X^2$ only at the point (a,a^2) as required.

FOR THE MORE INQUISITIVE STUDENT: You might try a similar approach to find the slope of the line tangent to the graph of $Y = X^3$ at the point (2, 8) or (a, a^3). The key idea is to find a value for m so that the choice of X = 2 (or a) will satisfy the equation of the curve and the line even after the factor X-2 (or X-a) has been eliminated. Thus if m is the slope of any line passing through (2,8), the line's equation will be Y = 8 + m(X - 2) and if this line has another point on the graph of $Y = X^3$ then $X^3 - 8 = m(X - 2)$.

Using some simple algebra we find that $(X^2 + 2X + 4 - m)(X - 2) = 0$.

This says that either X - 2 = 0 or $X^2 + 2X + 4 = m$. In this situation we characterize the line as tangent if X = 2 is a solution of this second equation. For X = 2 to satisfy this last equation it must be that m = 12. [When m = 12, X = 4 is also a solution! Is this okay?]

Review Exercises: The exercises for this section cover material which you may recall from previous course work in algebra and trigonometry. You may not recall all of these topics or how to do the problems precisely. You may want to refer to the texts or notes from your previous courses in mathematics if you find these difficult. The skills needed to solve these problems will be important in the work ahead- so be careful to identify any difficulties you have with these problems and try to remedy any misunderstandings as you proceed.

1. The line **L** goes through the point **(3,2)** with a **slope of -4.**

a. Give an equation for **L.**

b. What is the X intercept of **L**? [Recall the X intercept of a line is the point where the line crosses the X axis.]

c. What is the Y intercept of **L** ? [Recall the Y intercept of a line is the point where the line crosses the Y axis.]

2. When possible, find the slope, an equation and the X- and Y- intercepts of the line connecting each of the following pairs of points.
   a. (2,-2) (3,-1)
   b. (-4,1) (-3,-1)
   c. (3,5) (3,-2)
   d. (-5,2) (3,2)

3. The temperature on Wednesday morning was approximately linear in its relation to the time. At 8 A.M. it was 40 degrees fahrenheit and at 10 A.M. it was 64 degrees. Estimate the temperature on Wednesday morning at 8:40 A.M.
4. Verify that the line at the indicated point with the given slope is tangent to the graph of the indicated equation in the following senses:

i) the line meets the circle only at the indicated point and

ii) the X coordinate is a double (repeated) root of the equation that results by replacing Y in the circle's equation by the point-slope form of the line Y=m(X - a) + b.

   a. $X^2 + Y^2 = 25$ at (-4,3), m = 4/3. [For ii) you need to show that -4 is the only root of $X^2+(\frac{4}{3}(X+4)+3)^2 = 25$.]
   b. $Y = X^2$ at (2,4), m = 4. [Show 2 is the only root of $(4(X - 2) + 4) = X^2$]
   c. $Y = X^2$ at (a,a^2), m = 2a.
   d. $Y = X^3$ at (2,8), m = 12. [Show 2 is the repeated root of $(12(X - 2) + 8) = X^3$]
   e. $Y = X^3$ at (a,a^3), $m = 3a^2$.

5. Find the slope and an equation of the line tangent to the graph of the indicated equation at the indicated point:

   a. $X^2 + Y^2 = 25$ at (3,4).
   b. $X^2 + Y^2 = 25$ at (a,b).
   c. $Y = X^2$ at (5,25).
   d. $Y = X^3$ at (4, 64).
   e. $Y = X^4$ at (3,81).
   f. $Y = X^4$ at (a, a^4).

6. Find the slope of the line tangent to the graph of $Y = X^5$ at the point (a,a^5). Generalize this result and prove your conjecture for $Y = X^n$ using the technique of the last example.

7. Write a brief discussion of the relation between the slope of a line and its steepness. Be sure to include examples and illustrations.

8. Find the slope of the line tangent to the graph of $Y = 5X^2 + 3X + 7$ at the point (1,15) and then at the general point $(a, 5a^2 + 3a + 7)$.

9. Generalize the result of the last problem for $Y = AX^2 + BX + C$. Prove your conjecture.

10. The relation of the radius and the tangent to a circle at a given point on the circle is that they are perpendicular lines.

A line that is perpendicular to a tangent line is often called a **normal line** .

   a. Find an equation for the normal line to the graph of $Y = X^2$ at the point (3,9).
   b. Find an equation for the normal line to the graph of $Y = X^2$ at the point (a,a^2).

11. Find an equation for any lines tangent to the curve $Y = X^2$ that pass through the point (4,8). Write a brief description of your process in solving this problem. [Hint: What is an equation for a line that is tangent to $Y = X^2$?]

12. Suppose $Y = AX^2 + BX$.
   a. The slope of the line tangent to the graph of this equation at the point (1,4) is 7. Find the values of A, and B.
   b. Suppose the graph of the equation passes through the point (1,4). Find the values of A and B if the slope of the tangent line is i) 1 ; ii) 0 ; iii) -1 iv) -7.

13. Write a brief story in which a linear change in scales is needed to solve a problem. Find the linear equation from data related to your story.

14. Write a brief essay on how banks determine rates of exchange for foreign money.

Excursion into Lagrange Interpolation.

Two points in the plane determine a line, and therefore in coordinate geometry, the coordinates for two points determine the equation of a line. If we consider the case where the two points have different first coordinates, we can find a **linear function** the graph of which is the line determined by the two points. The method of Joseph Louis Lagrange, (1736-1813), called LaGrange interpolation resolves a comparable problem for determining a polynomial function of degree n the graph of which passes through n+1 points in the plane no two of which have the same first coordinate. In this excursion we'll look at the simplest cases when n= 1 and 2 to get a sense of how this method works, and leave the more general examples for the exercises. With a small number of points, this method is sometimes used to find a polynomial of low degree for estimating other information about a less well-known function.

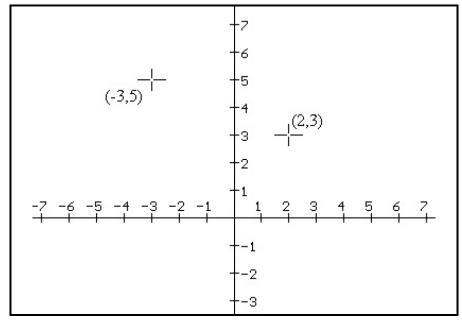

Figure 11

Example 1. We'll find the linear function that has its graph pass through two given points as an example of Lagrange's method. [Of course there are many methods you have studied for solving this problem. We are interested here on a method that will generalize for more points and polynomial functions of higher degree.] For this example, suppose the line passes through (2,3) and (-3,5). See Figures *** and ***.

For Lagrange's method we suppose the linear function has the form

$$f(X) = Y = R\ (X-2) + S\ (X+3).$$

Then using the coordinate information we replace the X and Y in the equation with X=2 and Y=3, giving us the equation 3= R(2-2) + S(2+3) or 3 = 5S, so S = 3/5. Similarly we can use X=-3, and Y=5, so 5 = -5R and thus R = -1. So the equation **f (X) =Y = -1(X-2) + 3/5 (X+3)** should work. You can check without much trouble that both the given points lie on the line this function determines. Notice how using factors of (X-2) and (X+3) in the form of the linear function made it easy to find the values for R and S.

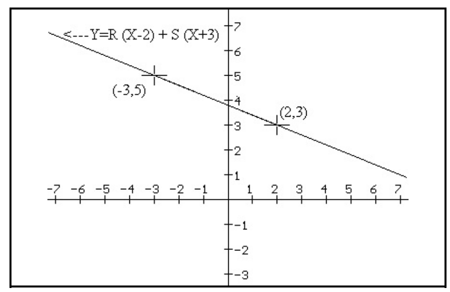

Figure 12

Example 2. We'll find the quadratic function describing a parabola that passes through three given points using Lagrange's method. For this example, suppose the parabola passes through (2,3), (-3,5), and (1,-7). See Figure ***. For Lagrange's method we suppose the quadratic function has the form

$$p(X) = Y = R\ (X-2)(X-1) + S\ (X+3)(X-1) + T\ (X-2)(X+3).$$

Using the coordinate information from the three given points we can determine R, S, and T. First we use X=2 andY=3 to see that 3=5S, so S = 3/5. Similarly using X=-3 and Y=5 we have 5=20 R, so R = 1/4 and with X = 1 and Y = -7 we find that -7=-4T, so T=7/4. So the equation for the quadratic function is

$$p(X)=Y = 1/4\ (X-2)(X-1) + 3/5\ (X+3)(X-1) + 7/4(X-2)(X+3).$$

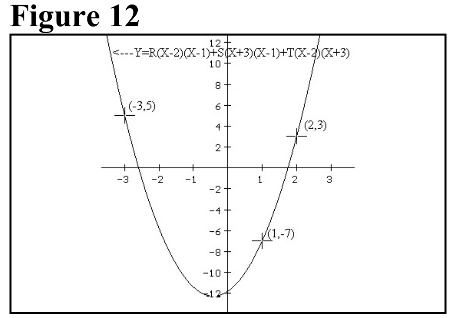

Figure 13

You can check this quadratic function does determine a parabola through the three given points. Notice how using factors of (X-2)(X-1), (X+3)(X-1), and (X-2)(X+3) in the form of the quadratic functions made it easy to find the values for R, S, and T.

15. Describe the key feature of the form of the linear function in Example 1 that enabled us to solve the problem.
    a. Suppose the problem in example 1 had points (2,1) and (-1,2). Use Lagrange's technique with f (X) = Y = R (X-a) + S (X-c) to find the equation for the appropriate linear function.

    b. Suppose the problem in example 1 had points (a,b) and (-c,d) with a$\neq$b. Use Lagrange's technique with f (X) = Y = R (X-a) + S (X-c) to show that the appropriate linear function is given by $f(X) = Y = d\,\dfrac{(X-a)}{(c-a)} + b\,\dfrac{(X-c)}{(a-c)}$.

16. Describe the key feature of the form of the quadratic polynomila function in Example 2 that enabled us to solve the problem.
    a. Suppose the problem in example 2 had points (1,2), (2,1), and (3,3). Use Lagrange's technique with $p(X)=Y=R(X-a)(X-e)+S(X-c)(X-e)+T(X-a)(X-c)$ to find the quadratic polynomial function with its graph passing through these three points.
    b. Suppose the problem in example 2 had points (1,2), (2,1), and (3,0). These three points all lie on the same line. Use Lagrange's technique with $p(X)=Y=R(X-a)(X-e)+S(X-c)(X-e)+T(X-a)(X-c)$ to show that the graph of no quadratic function could pass through these three points. [Hint: Suppose that there was such a quadratic and work out the values for R, S, and T. Then expand the polynomial and notice that R+S+T=0.]
    c. Suppose the problem in example 2 had points (a,b), (c,d), and (e,f) with a, c, and e all different numbers. Use Lagrange's technique with $p(X)=Y=R(X-a)(X-e)+S(X-c)(X-e)+T(X-a)(X-c)$ to show that the appropriate parabola has equation
    $$p(X) = Y = d\,\frac{(X-a)(X-e)}{(c-a)(c-e)} + b\,\frac{(X-c)(X-e)}{(a-c)(a-e)} + f\,\frac{(X-a)(X-c)}{(e-a)(e-c)}.$$

17. Use Lagrange interpolation to find an appropriate function passing through the given points. Sketch a graph of this function based only on the given points and what you think the curve must be. Compare your sketch with the graph created by graphing technology.
    a. A linear function passing through the points (-1,3) and (2,1).
    b. A quadratic function passing through the points (-1,3), (0,2), and (2,1).
    c. A cubic function passing through the points (-1,3), (0,2), (1,5), and (2,1).
    d. A quartic (fourth degree) polynomial function passing through (-2,4), (-1,3), (0,2), (1,5), and (2,1).

18. Finding a quadratic function that resembles other functions: By choosing three noncolinear points on any curve we can use Lagrange's interpolation to find a parabola that passes through those points. For each of the following functions find a parabola that passes through the graphs of the functions when at points with the indicated first coordinates. Use graphing technology to draw a sketch of the function and the quadratic function you find. Discuss how you might use the function you find to estimate the value of the given function.
    a. $f(x)=x^5-4x^3+2$; x = 0, 1, 2.
    b. $f(x)=\sqrt{x}$; x = 0, 1, 4.
    c. $f(x)=2^x$; x = -1, 0, 1.
    d. $f(x)=2^x$; x = 0, 1, 2.
    e. $f(x)=\sin(\frac{\pi}{2}x)$; x = 0, 1, 2.
    f. $f(x)=\cos(\frac{\pi}{2}x)$; x = -1, 0, 1.

19. Describe a general procedure for finding a polynomial function of degree n that passes through n+1 given points with distinct first coordinates.