Introduction to Sensible Calculus: A Thematic Approach Workshop
University of Utah
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The Sensible Calculus Program

- Primary Creator: Martin Flashman
  - Since 1981.
  - Associate: Tami Matsumoto (since 2006)
- Main Web Page (2002)
- Many materials currently available on-line through the main web page.
- Key content and pedagogical concepts will be the focus of this workshop.

Session I Calculus Mapping Figures

We begin our introduction to mapping figures by a consideration of linear functions :

" $y = f(x) = mx + b$ "

Mapping Figures

A.k.a.
Function diagrams
Transformation Figures
Dynagraphs

Written by Howard Swann and John Johnson

A early source for visualizing functions at an elementary level before calculus.

YOUR FRIENDLY NEIGHBORHOOD FUNCTION CONSISTS OF TWO SETS AND A BUNCH OF ARROWS THAT OBEY

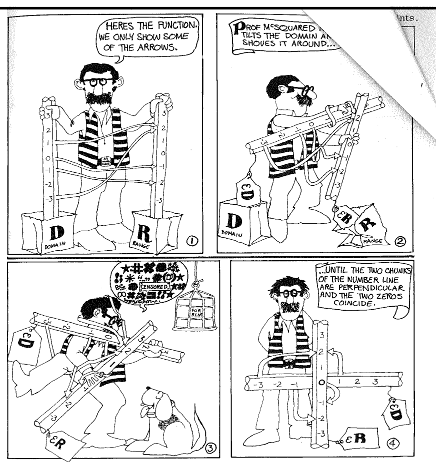
RULE **1** THE ARROWS ALWAYS START FROM THE SAME SET, CALLED THE **DOMAIN** AND GO TO THE OTHER SET, CALLED THE **RANGE**.

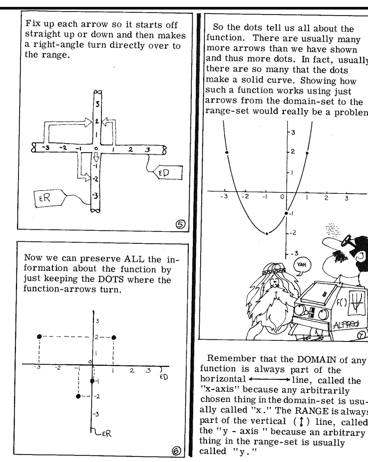
RULE **2** EVERYTHING IN THE DOMAIN-SET MUST HAVE EXACTLY ONE ARROW FROM IT. EVERYTHING IN THE RANGE-SET MUST HAVE AT LEAST ONE ARROW TO IT.

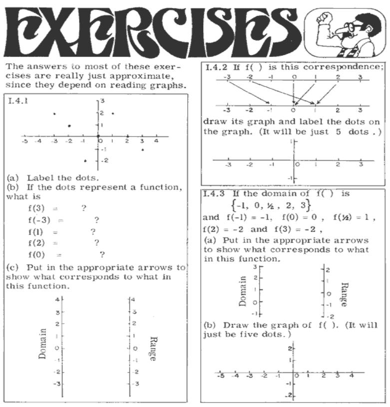
(IT'S OK TO HAVE 2 ARROWS TO 1 THING.)

So two or more arrows can hit the same thing in the range-set, but only one arrow can come from any particular thing in the domain-set.

Using arrows in the RULES unfortunately has its drawbacks —as functions become more elaborate, the arrows can get pretty difficult to follow . . . .







EXERCISES

The answers to most of these exercises are really just approximate, since they depend on reading graphs.

I.4.1

(a) Label the dots.
(b) If the dots represent a function, what is

f(3) = ?
f(-3) = ?
f(1) = ?
f(2) = ?
f(0) = ?

(c) Put in the appropriate arrows to show what corresponds to what in this function.

Domain

Range

I.4.2 If f( ) is this correspondence;

draw its graph and label the dots on the graph. (It will be just 5 dots.)

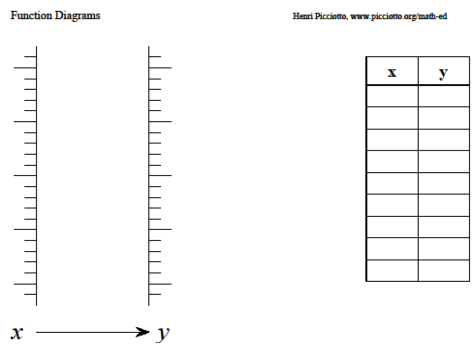
I.4.3 If the domain of f( ) is {-1, 0, ½, 2, 3}
and f(-1) = -1, f(0) = 0, f(½) = 1, f(2) = -2 and f(3) = -2,
(a) Put in the appropriate arrows to show what corresponds to what in this function.

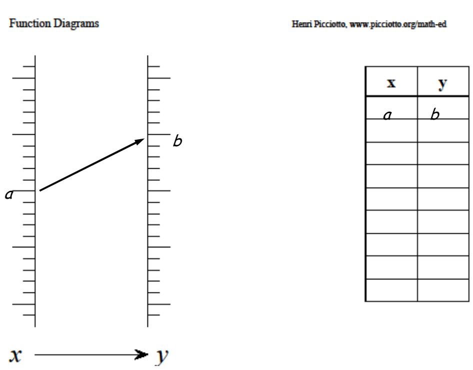
Domain

Range

(b) Draw the graph of f( ). (It will just be five dots.)

Function Diagrams by Henri Picciotto





Mapping Diagrams and Functions

- SparkNotes › Math Study Guides › Algebra II: Functions Traditional treatment.
  - http://www.sparknotes.com/math/algebra2/functions/
- **Function Diagrams. by Henri Picciotto Excellent Resources!**
  - **Henri Picciotto's Math Education Page**
  - **Some rights reserved**
- Flashman, Yanosko, Kim https://www.math.duke.edu//education/prep02/teams/prep-12/

Ideas

- Interpretation of axes
- Linear
- Nonlinear (quadratic, trig, exp/ln, 1/x)
- inverses
- Composition
- Derivative (visualize rate/ratio)
- Differential

Linear Functions: Tables

x	5 x - 7
3	
2	
1	
0	
-1	
-2	
-3	

Complete the table.
x = 3,2,1,0,-1,-2,-3
f(x) = 5x – 7

f(0) = ____?

For which x is f(x) > 0?

Linear Functions: Tables

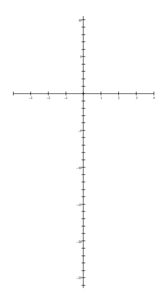
X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Complete the table.
x = 3,2,1,0,-1,-2,-3
f(x) = 5x – 7

f(0) = ____?

For which x is f(x) > 0?

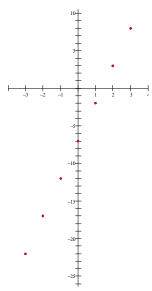
Linear Functions: On Graph



Plot Points (x , 5x - 7):

X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

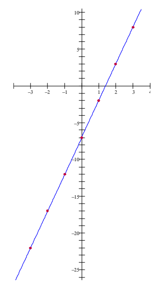
Linear Functions: On Graph



Connect Points
(x , 5x - 7):

X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Linear Functions: On Graph



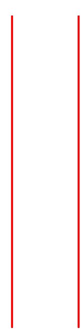
Connect the Points

X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Linear Functions: Mapping Figures

- Connect point x to point 5x – 7 on axes

X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22



Linear Functions: Mapping Figures

X	5 x - 7
3	8
2	3
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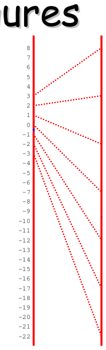


Sensible Calculus Resources

- **Visualizations and Transformation Figures**
- Ch 0.B.2 Functions-Introduction and Review.

Examples on Excel, Winplot, Geogebra

- Excel example:
- Winplot examples:
  - Linear Mapping examples
- Geogebra examples:
  - dynagraphs.ggb
- Web links:
  - https://www.math.duke.edu//education/prep02/teams/prep-12/
  - http://www.dynamicgeometry.com/JavaSketchpad/Gallery/Trigonometry_and_Analytic_Geometry/Dynagraphs.html
  - http://demonstrations.wolfram.com/Dynagraphs/
  - http://demonstrations.wolfram.com/ComposingFunctionsUsingDynagraphs/
  - http://users.humboldt.edu/flashman/TFLINX.HTM

Simple Examples are important!

f(x) = mx + b with a mapping figure -- Five examples:

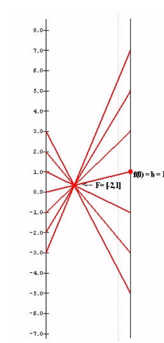
- Example 1: m =-2; b = 1: f(x) = -2x + 1
- Example 2: m = 2; b = 1: f(x) = 2x + 1
- Example 3: m = ½; b = 1: f(x) = ½ x + 1
- Example 4: m = 0; b = 1: f(x) = 0 x + 1
- Example 5: m = 1; b = 1: f(x) = x + 1

Visualizing $f(x) = \boldsymbol{mx} + \boldsymbol{b}$ with a mapping figure -- Five examples:

Example 1: m = -2; b = 1
f (x) = **-2x + 1**

- Each arrow passes through a single point, which is labeled **F = [- 2,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $-2x+1$,

    which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

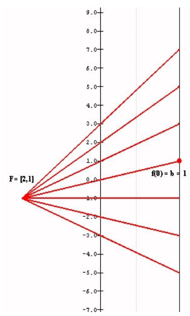
Example 2: $m = 2;\ b = 1$
f(x) = 2x + 1

Each arrow passes through a single point, which is labeled

F = [2,1].

- The point **F** completely determines the function *f*.
  - **given a point** / number, *x*, on the source line,
  - there is a **unique arrow passing through F**
  - **meeting** the target line at **a unique point** / number, $2x + 1$,

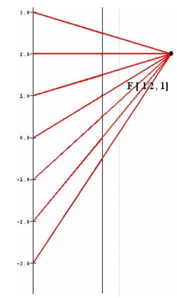
  which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

- **Example 3:** $m = 1/2;\ b = 1$
  f(x) = ½ x + 1
- Each arrow passes through a single point, which is labeled **F = [1/2,1].**
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, ½ x + 1,

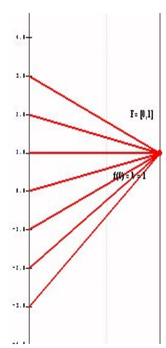
    which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

- **Example 4:** $m = 0;\ b = 1$
  f(x) = 0 x + 1
- Each arrow passes through a single point, which is labeled **F = [0,1].**
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $f(x)=1$,

    which corresponds to the linear function's value for the point/number, *x*.



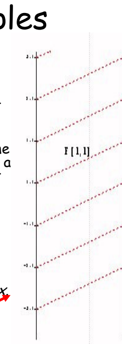
Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples

Example 5: $m = 1;\ b = 1$

$f(x) = x + 1$

- Unlike the previous examples, in this case it is not a single point that determines the mapping figure, but the single arrow from 0 to 1, which we designate as F[1,1]
- It can also be shown that this single arrow completely determines the function.Thus, given a point / number, *x*, on the source line, there is a unique arrow passing through *x* ***parallel to*** F[1,1] meeting the target line a unique point / number, $x + 1$, which corresponds to the linear function's value for the point/number, *x*.
  - The single arrow completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow** through *x* ***parallel to*** F[1,1]
    - **meeting** the target line at **a unique point** / number, $x + 1$,

which corresponds to the linear function's value for the point/number, *x*.



Mapping Figures and The Derivative

Motivation and Balance Estimation and Local Linearity

The Sensible Calculus Program Introduction to the Derivative

- Motivation for the derivative as a number, visual and numerical estimation with graphs and mapping figures.
- Ch 0 A Motivation: What is the calculus?
- Ch 0 B Solving the Tangent Problem
- Ch 1 A Tangent Line
- Ch 1 B Velocity
- Ch 1 D Derivative (Four Steps)

End of Session I

- Questions
- Break - food and thought
- Partner/group integration task

Morning Break: Think about These Problems

M.1 Use a **mapping figure for the function** $f(x) = -3x + 2$ **to illustrate that** $f'(1) = -3$.

Sketch a mapping figure that illustrates the work to show that **the linear function** $f(x) = mx + b$ has $f'(a) = m$. **Discuss how different values of m impact your figure.**

M.2 Use a **mapping figure for the function** $f(x) = x^2$ **to illustrate that** $f'(3) = 6$.
Sketch a mapping figure that illustrates the work to show that $f'(a) = 2a$.

M.3 Use a **mapping figure for the function** $f(x) = 1/x$ **to illustrate that** $f'(2) = -1/4$.
Sketch a mapping figure that illustrates the work to show that $f'(a) = -1/a^2$.

Session II Differential Equations, Approximation and The Fundamental Theorem of Calculus

We continue our introduction to A Sensible Calculus by a consideration of the **FT of Calculus from a sensible view of DE's and estimations using Euler's Method interpreted in as variety of contexts.**

The Sensible Calculus Program

The **FT of Calculus, DE's, and Euler's Method**

- Motivation for the FT of C from estimating a solution to an Initial Value Problem, visual and numerical estimation with graphs and mapping figures. Sensible and balanced interpretation the FT of C and integrals.
- **Ch III.A.1. THE DIFFERENTIAL**
- **Comment on the Mean Value Theorem, Implicit Differentiation, Related Rate, and "graphing" Applications being connected to DE's**
- **Ch IV Differential Equations from an Elementary Viewpoint**
- V.A The Definite Integral

Conversion ☺

- Discuss how to convert the following exercises to visualize the motivating problem with mapping figures. Discussion: What are the advantages? Disadvantages?
- Introduction to Integration - The Exercise Bicycle Problem: Part 1 Part 2 by Marc Renault, Shippensburg University.

End of Session II

- Questions
- Lunch Break - food and thought
- Partner/group integration task

Lunch Break: Think about These Problems

L.1 **Assume y is a solution to the differential equation**

$$dy/dx = 1/(x^2 + 1) \text{ with } y(0)=2.$$

(a) Using just the given information, find any local extreme points for y and discuss the graph of y, including the issue of concavity.
(b) Using the differential, estimate $y(1)$ and $y(-1)$.

L.2 **Assume y is a solution to the differential equation**

$$dy/dx = 1/(x^2 + 1)$$

(a) Sketch the tangent field showing tangents in all four quadrants.
(b) Draw three integral curves on your sketch **including one through the point (1,2)**;
(c) Suppose that a solution to the differential equation has value 2 at 1.
  (i) Based on your graph, **estimate** the value of that solution at 2.
  (ii) Estimate the value of y(3) using Euler's method with n = 4.

L.3 **Assume y is a solution to the differential equation**

$$dy/dx = -y/x$$

(a) Sketch the tangent field showing tangents in all four quadrants.
(b) Draw three integral curves on your sketch **including one through the point (1,2)**;
(c) Suppose that a solution to the differential equation has value 2 at 1.
  (i) Based on your graph, **estimate** the value of that solution at 2.
  (ii) Estimate the value of y(2) using Euler's method with n = 4.

L.4 Suppose $y'' = -y$, $y'(0) = 1$ and $y(0) = 0$. Estimate y(1), y(2), y(3), and y(4).

Session III More on DE's and Estimations Pedagogy: Fundamental Concepts, not "foundations".

We continue our introduction to Sensible Calculus by considering the pedagogy for conceptual approaches. Models and connecting with the familiar.

The Sensible Calculus Program

More on DE's, Models and Estimations

- Modeling contexts provide a sensible source for for DE's and both practical and theoretical use of concepts and skills. Pedagogical decisions are made to stay focused on the themes by engaging students in balanced approaches using visual, symbolic, numerical and verbal approaches.
- VI.A Differential Equations & Models- The Exponential Function
- VI.B Differential Equations & Models- The Natural Logarithm Function
- VI.C Connecting the Natural Logarithm and Exponential Functions
- VI.D More Models & Inverse Trigonometry
- IXA Taylor Theory for e^x

"Pedagogical" Principles

- Themes of differential equations and estimation throughout the first year of calculus, using modeling as a central motivation for applications of the calculus.
- "…*everything in a calculus course can be related to the study of differential equations.* "
- "…*estimation is valuable for both numerical and conceptual development.* "
- The consistent use of interpretations to provide meaning for calculus concepts.
- "… *models as sources for concepts and interpretations as well as for applications."*
- Present examples of models or arguments before more general applications and proofs.

- Informal understanding habits form a learning foundation for later concept, language, and notation definitions.
- Students can understand the specific and particular in experience and then generalize more easily then they can understand a general proposition or proof and the apply it to the particular.
- If a topic is sensibly organized by itself and sensibly placed with regard to the other topics, then it should remain a part of the course. But if it fails to make sense locally or globally, it needs careful reassessment and revision.

End of Session III

- Questions
- Break - food and thought
- Partner/group integration task

Session IV Amazing Results: Integral of exp(-x^2) Newton's Estimate of ln(2)

We complete our introduction to the Sensible calculus by a consideration of **some amazing results that provide both motivation and consolidation for the first year experience with calculus.**

Amazing Results

- How Newton used Geometric series to find ln(2)
- Finding $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

Session V Mapping Figures and Technology

Technological support for the Sensible Calculus… What do you need?
Winplot
Excel
Graphing calculators
Geogebra and GSP
Mathematica, etc.

Examples on Excel, Winplot, Geogebra

- **Excel example(s):**
  - **Mapping figures and graphs**
  - **Euler's Method**
  - **Integral Estimates**
- **Winplot examples:**
  - **Linear Mapping examples**
  - **Direction Fields; Euler's Method**
  - **Integral Estimates**
  - **Taylor Theory**
- Geogebra examples:
  - geogebratube
  - dynagraphs.ggb
  - Composition.ggb
- Other Web links:
  - https://www.math.duke.edu//education/prep02/teams/prep-12/
  - http://demonstrations.wolfram.com/topics.html?Mathematics#2
  - http://www.wolframalpha.com

Thanks The End!

☺

Questions?

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- Flashman, Yanosko, Kim https://www.math.duke.edu//education/prep02/teams/prep-12/

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- [FL3] Flashman, Martin. "Concepts to Drive Technology," in Proceedings of the Fifth Annual Conference on Technology in Collegiate Mathematics, November 12-15, 1992, edited by Lewis Lum, Addison-Wesley, 1994.
- [FL4] Flashman, Martin. "Historical Motivation for a Calculus Course: Barrow's Theorem," in Vita Mathematica: Historical Research and Integration with Teaching, edited by Ronald Calinger, MAA Notes, No. 40, 1996.