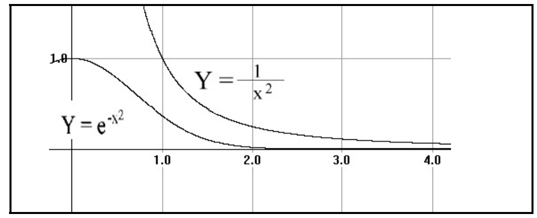
The Integral of e^{-x^2} over All Real Numbers

We'll denote the integral of e^{-x^2} over all real numbers by the letter **I**. That is, $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.

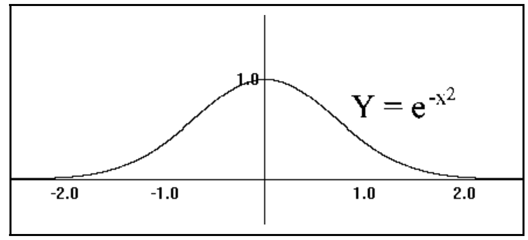
First we claim that **I is convergent:**



For any $x > 1$, $e^{-x^2} < 1/x^2$, and

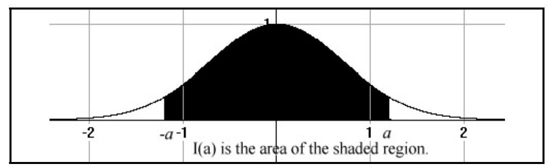
$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{k\to\infty} -\frac{1}{x}\Big|_1^k = \lim_{k\to\infty} 1 - \frac{1}{k} = 1$$

. Using the comparison test then we can see that $\int_1^{\infty} e^{-x^2}\,dx$ converges and hence that $\int_0^{\infty} e^{-x^2}\,dx$ converges as well. Furthermore the graph of e^{-x^2} is symmetric with respect to the Y-axis and so, using this symmetry, we have that $\int_{-\infty}^{0} e^{-x^2}\,dx$ is a convergent integral. Consequently



$I = \int_{-\infty}^{0} e^{-x^2}\,dx + \int_0^{\infty} e^{-x^2}\,dx$ is a convergent integral.

We can estimate **I** numerically with a symmetric integral which we denote as $I(a) = \int_{-a}^{a} e^{-x^2}\,dx$. For example, using technology to make an estimate you should be able to find that I(10) ≈ 1.772455. This estimate is very close to estimates with much higher values for a. The graph in Figure 3 illustrates this integral as an area.



We notice at this stage something remarkable.... $I(10)^2$ is approximately 3.141597, very close to π !

We'll show that $I^2 = \pi$, and therefore that $I = \sqrt{\pi}$.

We begin the demonstration by rotating the graph of e^{-x^2} in the first quadrant around the "Y axis". [See the graph of this surface in Figure 4.]

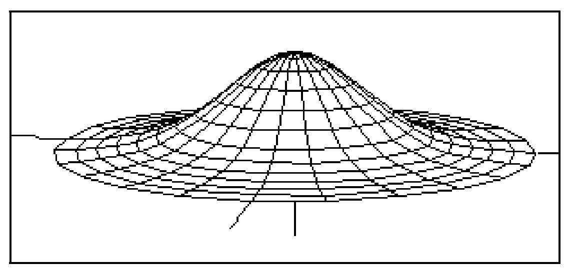
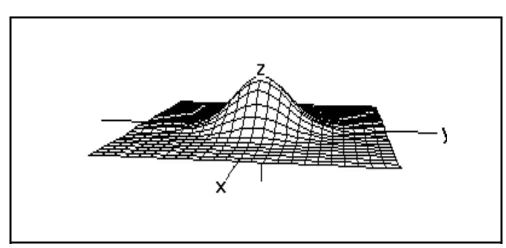


Figure 4

Now we'll analyze the volume of the region enclosed below this surface and above the plane by the method of "shells." In fact, we can find the volume under this surface over the entire plane! This is a limit, namely, $\lim_{k\to\infty} \int_0^{K} 2\pi r e^{-r^2}\,dr$. A numerical estimate for this infinite integral, using K = 10, is $\int_0^{10} 2\pi r e^{-r^2}\,dr \approx 3.141593$. The actual integral is found easily using the substitution $u = -r^2$ and the fundamental theorem of calculus.

We see that $\int_0^{K} 2\pi r e^{-r^2}\,dr = \pi(1 - e^{-k^2})$ which approaches π as k → ∞. Aha! π again!



Now here's the key to our investigation of **I**. We'll find this same volume using cross sections perpendicular to the X- axis. Another graph of the same surface now organized for cross-section analysis should be drawn here. Note each cross section is a scalar multiple of the graph of e^{-x^2}. This scalar at $x = a$ is precisely e^{-a^2}.

Some examples here may help convince you , so draw the graphs of $e^{0} e^{-y^2}$, $e^{-(\frac{1}{2})^2} e^{-y^2}$, $e^{-1} e^{-y^2}$, and $e^{-2^2} e^{-y^2}$. More precisely the height of the surface above the point with coordinate (a,y) will be $e^{-(a^2+y^2)} = e^{-a^2} e^{-y^2}$. Thus the area of each cross section of the region with X=a enclosed by our surface is precisely $\int_{-\infty}^{\infty} e^{-a^2} e^{-y^2} dy = e^{-a^2} \int_{-\infty}^{\infty} e^{-y^2} dy = e^{-a^2} I$ where **I** is the integral we are trying to find.

The volume under the entire surface is $\lim_{k \to \infty} \int_{-K}^{K} e^{-x^2} * I \, dx = I * \lim_{k \to \infty} \int_{-K}^{K} e^{-x^2} dx = I^2$

THEREFORE THE VOLUME OF THE REGION IS $\mathbf{I}^2$.

But we computed the volume already to be π. Therefore $\mathbf{I}^2 = \pi$ and $I = \sqrt{\pi}$.

Exercises: Evaluate the following integrals using the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

1. $\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx$. [Hint: Use a substitution, $u^2 = x^2/2$.]

2. $\int_{0}^{\infty} e^{\frac{-x^2}{2}} dx$.

3. Show that $f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2}}$ is a probability density function over all the real numbers.