The Benefits of A Habit: Examining Evidence to Understand Statements and Proofs.

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Acknowledgement

This work is based in part on

experiences teaching at HSU: Math 240 (3 units) **Introduction to Mathematical Thought and Math 381 (1 unit) Tutorial in Writing Proofs**

Using

Daniel Solow's **How To Read and Do Proofs (Solow 1) and**

The Keys to Advanced Mathematics... (Solow 2)

and ***How To Solve It*** by G. Polya.

Initial Thoughts

- Polya’s 4 Phases of Problem Solving

from “How To Solve It" by G. Polya**.**

1. Understand the problem.
2. See connections to devise a plan.
3. Carry out the plan.
4. Look back. Reflect on the process and results.

Taking Steps Too Fast

- Example:
- If n is an odd integer then $1+ n^2$ is even.
- Step 1: Assume we understand the problem – so
- Step 2: Make a plan. (Using Solow 1 or 2)

Since this is a conditional statement-

Assume the hypothesis, - n is an odd integer…

then consider a key question for conclusion,
  How do I show a number is even? … So …

Work backwards- Enough to show:
  There is some integer k where $2k=1+ n^2$

Work forward: Do calculations with $n= 2m+1$.

Taking Steps Too Fast

- Step 3: Execute the plan.

Assume n is odd… n=2m+1…

$1+ n^2 = 1 + (1+ 2m)^2 = 2+4m +4m^2 \quad = 2\ k$

Where $k = 1+2m +2m^2$ … EOP.

- Step 4: Reflect on process and result.

Why does this seem so easy to mathematicians yet so difficult for some of our students?

The faulty assumption.

- Example:
- If n is an odd integer then $1+ n^2$ is even.
- Step 1: Assume we understand the problem.
- Some students read the statement- recognize it is a conditional, but have not understood the content meaning in its context.
- They do not have the habit of thought to **attach interpretations to the words in the context** that prepares them to proceed to the next step and use the format of the problem to devise a plan.

What habit?

- Mathematicians have a habit of thought in **giving statements meaning in contexts**.
- This habit allows the mathematician to **understand initially a statement's interpretation** and focus attention on **the material meaning** in a sentential or quantified form.
- Without this habit, a student's initial interpretation statement **may lack an actual understanding of the content meaning of the statement in an appropriate context**.
- Without this understanding, the student can proceed too quickly to making a plan and discover confusion from being inadequately prepared.

Developing a habit.

- Working pedagogical hypothesis:
  Students can benefit in their early steps in learning to compose a proof by developing ***a habit of examining evidence*** **to attach meaning to the words in the problem before working on a plan.**
- Assumption: A habit of examining evidence can develop through small and regular repetition with eventual "rewards."
  [Drill and kill???]

Karate Kid Approach
"Drill and Kill ???"

Sacred Pact: (Mr. Miyagi)
"I promise teach Karate. You promise learn. I say- you do. No questions"

Application:

"First wash all the car then wax…. wax on - wax off. **Breath** in through nose, out the mouth"

Sand the floor:… "right a circle, left a circle. **Breath** in, breath out"

"Paint the fence. All in the wrist… Wrist up, wrist down. Don't forget to **breath**."

See http://www.youtube.com/watch?v=8aYl7N0JPWs

[redacted]

Examples of work intended to develop a habit.
Guess and Check vs. Context based reason/algorithm

Problem 2:

(i) Solve the equations (Show your work and check your answer.) :

a. $3x=15$

Work:

Solution:

Check:

b. $3x=24$

Work:

Solution:

Check:

c. $3x=7$

Work:

Solution:

Check:

d. $3x=N$

Work:

Solution:

Check:

Some examples of work intended to develop a habit.
Use of algorithmic processing.

(ii) Find the following integrals. (Show your work and check your answer.)

a. $\int_0^1 x^2 dx$

Work:

Solution:

b. $\int_0^1 x^3 dx$

Work:

Solution:

c. $\int_0^1 x^{1/2} dx$

Work:

Solution:

d. $\int_0^1 x^N dx \quad N \neq -1$

Work:

Solution:

e. Discuss briefly how you can justify your solutions as being correct.

Some examples of work intended to develop a habit. Examining evidence to attach meaning to the words.

Suppose X = { 1, 3, 5, a, c, e}, Y = { 1, 2, 3, a, b, c }, Z = { 2, 4, 6, b, d, f } and W={4, 5, 6, d, e, f }

a. Is $1 \in$ X?_______ Y? ______ Z ? ______ W? _______

b. Is $2 \in$ X?_______ Y? ______ Z ? ______ W? _______

c. Is $a \in$ X?_______ Y? ______ Z ? ______ W? _______

d. Is $b \in$ X?_______ Y? ______ Z ? ______ W? _______

e. List all sets that have 3 as an element. ____________________

f. List all elements that are **members of both X and Y.** ___________

g. List all elements that are **members of either X or Y.** ____________

h. List all elements of **X that are not elements of Y.** ____________

i. List all elements of the set **Z** $\cap$ **W.**______________________

j. List all elements of the set **Z** $\cup$ **W.** _______________________

k. List all elements of the set **Z - W.**________________________

More examples of work intended to develop a habit. Examining evidence to attach meaning to the words.

Problem 2: Consider the following intervals of real numbers: X = (1, 5], Y = [1,3], Z = [2, 6) and W= (4,5].

a. Is $1 \in X$?_______ Y? ______ Z ? ______ W? _______

b. Is $2 \in X$?_______ Y? ______ Z ? ______ W? _______

c. Is $3 \in X$?_______ Y? ______ Z ? ______ W? _______

d. Is $4 \in X$?_______ Y? ______ Z ? ______ W? _______

e. List all sets that have 3 as an element. ____________________

f. Complete the following description of all elements that are **members of both X and Y.**

Any element of both X and Y is a real number that ________________________________

g. Complete the following description of all elements that are **members of either X or Y.**

Any element of either X or Y is a real number that ________________________________

h. Complete the following description of all elements of **X that are not members of Y.**

Any element of X that is not a member of Y is a real number that ________________________

i. Complete the following description of all elements of the set **$Z \cap W$.**

Any member of **$Z \cap W$** is a real number that ________________________________

j. Complete the following description of all elements of the set **$Z \cup W$.**

Any member of **$Z \cup W$** is a real number that ________________________________

Some examples of work intended to develop a habit. Examining evidence to attach meaning to the words.

Suppose X ={All circles with a radius of length r meters where $1< r \le 5$} ,
Y ={All circles with a radius of length r meters where $1\le r \le 3$}
Z ={All circles with a radius of length r meters where where $2 \le r <6$}
W ={All circles with a radius of length r meters where $4< r \le 5$ }

a. Is a circle of radius 1 meter $\in$ X?________ Y? _______ Z ? ________ W? ________

b. Is a circle of radius 2 meters $\in$ X?________ Y? _______ Z ? ________ W? ________

c. Is a circle of radius 3 meters $\in$ X?________ Y? _______ Z ? ________ W? ________

d. Is a circle of radius 4 meters $\in$ X?________ Y? _______ Z ? ________ W? ________

e. List all sets that have a circle of radius 3 meters as an element. __________________

f. Describe all circles that are **members of both X and Y.**

Any element of both X and Y is a ____

g. Describe all circles that are **members of either X or Y.**

Any element of either X or Y is a ____

h. Describe all circles that are elements of **X that are not members of Y.**

Any member of X that is not a member of Y is a ________________________________

i. Describe all circles that are elements of the set $\mathbf{Z \cap W}$.

Any element of $\mathbf{Z \cap W}$ is a ____

j. Describe all circles that are elements of the set $\mathbf{Z \cup W}$.

Any member of $\mathbf{Z \cup W}$ is a ____

k. Describe all circles that are elements of the set **Z - W.**

Any member of **Z - W** is a ____

Some examples of work intended to develop a habit. Expressing "key questions".

For each of the following statements give **a key question** for showing that the unquantified **statement is true. Answer your key question.**

a. For all x, x^2 is a member of E .

Key question:

Answer:

b. There is an x so that x^2 is a member of O

Key question:

Answer:

c. There is an x in T with x^2 is a member of T.

Key question:

Answer:

d. For all x in E, x is a member of $E \cap T$.

Key question:

Answer:

e. For all x in E, x is a member of $E \cup T$.

Key question:

Answer:

f. Some a member of E is a member of E-T.

Key question:

Answer:

Some examples of work intended to develop a habit. <u>Understanding</u> component meanings.

E = { n: n is an integer and there is an integer k where $n = 2k$};
O = { n: n is an integer and there is an integer k where $n = 2k + 1$}
T = { n: n is an integer and there is an integer k where $n = 3k$}
For each of the following conditional statements if possible **give separate examples of a number x (i) where the hypothesis is true; (ii) where the conclusion is true; (iii) where the hypothesis is false; (iv) where the conclusion is false.**
Do you believe the conditional statement is true or false?

a. If x is a member of E then x^2 is a member of E

.

b. If x is a member of O then x^2 is a member of O .

c. If x is a member of T then x^2 is a member of T.

d. If x is a member of O then $x+1$ is a member of E .

e. If x is a member of T then $x + 1$is a member of E.

f. If x is a member of T then $x^2 + x$ is a member of E .

Some examples of work intended to develop a habit. Understanding component meanings.

Recall
E∩T = { x: x is a member of E **and** x is a member of T} [intersection];
E∪T = { x: x is a member of E **or** x is a member of T} [union]
E-T = { x: x is a member of E and x is **not** a member of T} [difference... complement]

For each of the following conditional statements if possible **give separate examples of a number x (i) where the hypothesis is true; (ii) where the conclusion is true; (iii) where the hypothesis is false; (iv) where the conclusion is false.**
Do you believe the conditional statement is true or false?

a. If x is a member of E then x is a member of E∩T.

b. If x is a member of E∩T then x is a member of E.

c. If x is a member of E then x is a member of E∪T .

d. If x is a member of E∪T then x is a member of E.

e. If x is a member of E then x is a member of E-T.

Some examples of work intended to develop a habit. Understanding evidence in negation.

For each of the following statements :

i) State its negation.

ii) Give an example of a value for x when the negation is true.

iii) Give an example of a value for x when the negation is false.

a. x^2 is a member of E .

b. x^2 is a member of O.

c. x^2 is a member of T.

d. x is a member of $E \cap T$.

e. x is a member of $E \cup T$.

Some examples of work intended to develop a habit. Understanding component meanings.

For each of the following statements if possible **give separate examples of a number *x* (i) where the unquantified statement is true; (ii) where the unquantified statement is false.**
Do you believe the quantified statement is true or false?

a. For all x, x^2 is a member of E .

b. There is an x so that x^2 is a member of O .

c. There is an x in T with x^2 is a member of T.

d. For all x *in* O, $x+1$ is a member of E .

e. There is an x in T with $x + 1$is a member of E.

f. For all x, $x^2 + x$ is a member of E .

Some examples of work intended to develop a habit. Understanding component meanings.

For each of the following statements if possible **give separate examples of a number *x* (i) where the unquantified statement is true; (ii) where the unquantified statement is false.**
Do you believe the quantified statement is true or false?

a. For all x in E, x is a member of $E \cap T$.

b. There is a member of $E \cap T$ that is a member of E.

c. For all x in E, x is a member of $E \cup T$.

d. There is a member of $E \cup T$ that is a member of E.

e. Some a member of E is a member of E-T.

f. Every member of E-T is a member of E.

g. Every member of T is a member of E-T.

Some examples of work intended to illustrate how the previous exercises can be put together.

For each of the following statements , **decide whether it is true or false.**

If false, give an example [a counterexample] to show it is false.

If true, give a proof or proof outline. Be sure to be clear whether you are moving forwards or backwards. Place any key questions you are using to plan your work inside brackets. [Key question: How do I/we show that.... ?]

1. .For all x, x^2 is a member of E . True or False?___________

2. There is an x so that x^2 is a member of O. True or False?___________

3. If x is a member of T then x^2 is a member of T. True or False?___________

4. For all x in E, x is a member of $E \cup T$. True or False?___________

5. For all x in E, x is a member of $E \cap T$. True or False?___________

6. Some member of E is a member of E-T. True or False?___________

7. If x is a member of $E \cap T$, then there is an integer k where $x = 6k$. True or False?___________

References

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Pólya, George. *How to Solve It*. (Doubleday, 1957).

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The End

- Questions?
- Comments?
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