Projection: Examples to Illuminate Unification, Generalization, and Abstraction.

HSU Mathematics Department Colloquium

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Abstract

- The concept of projection has been important in understanding knowledge at least since Plato's Cave. In this presentation Professor Flashman will use projection to enrich understanding of the mathematical processes of unification, generalization, and abstraction. Examples will be given showing how projection and these processes add to the tools for computation and proof in a variety of contexts.

Caveat

- This presentation has a mixture of thoughts I have organized based on ponderings about
  - Pedagogy: How to teach proof writing and mathematical Thought.
  - Philosophy: How to understand what is mathematics (ontology) and how we come to mathematical knowledge (epistemology).
  - Mathematics: How to connect the mathematics of projection.
- There is a hope – but no guarantee - that this presentation will shed light on these separate but related human interests.

What is “Generalization” Polya on....

- “ … passing from the consideration of one object to the consideration of a set containing that object;
- or passing from the consideration of a restricted set to that of a more comprehensive set containing the restricted one.”

Refining the concept of "generalization" (Solow)

Daniel Solow describes three "uses of mathematics":

1. Unification
2. Generalization
3. Abstraction

Unification [Solow]

Unification: “The process of creating …[a] new, encompassing representation… . This technique involves combining two or more concepts (problems, theories, and so on) into a single framework from which you can study each of the special cases.”

Generalization [Solow]

"**Generalization** is the process of creating, from an original concept (problem, definition, theorem, and so on), a more general concept (problem, definition, theorem, and so on) that includes not only the original one, but many other new ones as well."

Abstraction [Solow]

"**Abstraction** is the process of taking the focus farther and farther away from specific items by working with general *objects*."

Context

It is important to understand the role that **context** plays with these three uses in mathematical thought.

Comment: Context was a key element of Bertrand Russell's analysis of the use of language in "On Denoting".
It is an important element in understanding interpretations in foundations and model theory.

Context Connections

- **U**nification: A single context.
- **G**eneralization: Distinct but connected, related contexts.
- **A**bstraction: Broad context definition with a structural characterization that allows discourse with limited specificity.

Approach to Proof and Problem Solving (Polya expanded)

Polya steps in light of Context and organization

1. **Understand the context** of the statement(s).
   - Is this a **unification,**
   - **generalization, or**
   - **abstraction?**
2. **R**ecognize how to connect understanding of instances and the context to **develop a plan** with an appropriate **organization**.
3. **Execute the plan** at the appropriate level of discourse and with a clear **organization**.
4. **Reflect on the role of context and how that effected the plan and its execution.**

Comment

- Most of the examples I will present in this presentation will relate to unification and generalization, a few will pertain to abstraction.

"Projection"! Plato: Inside the cave [from wikipedia]

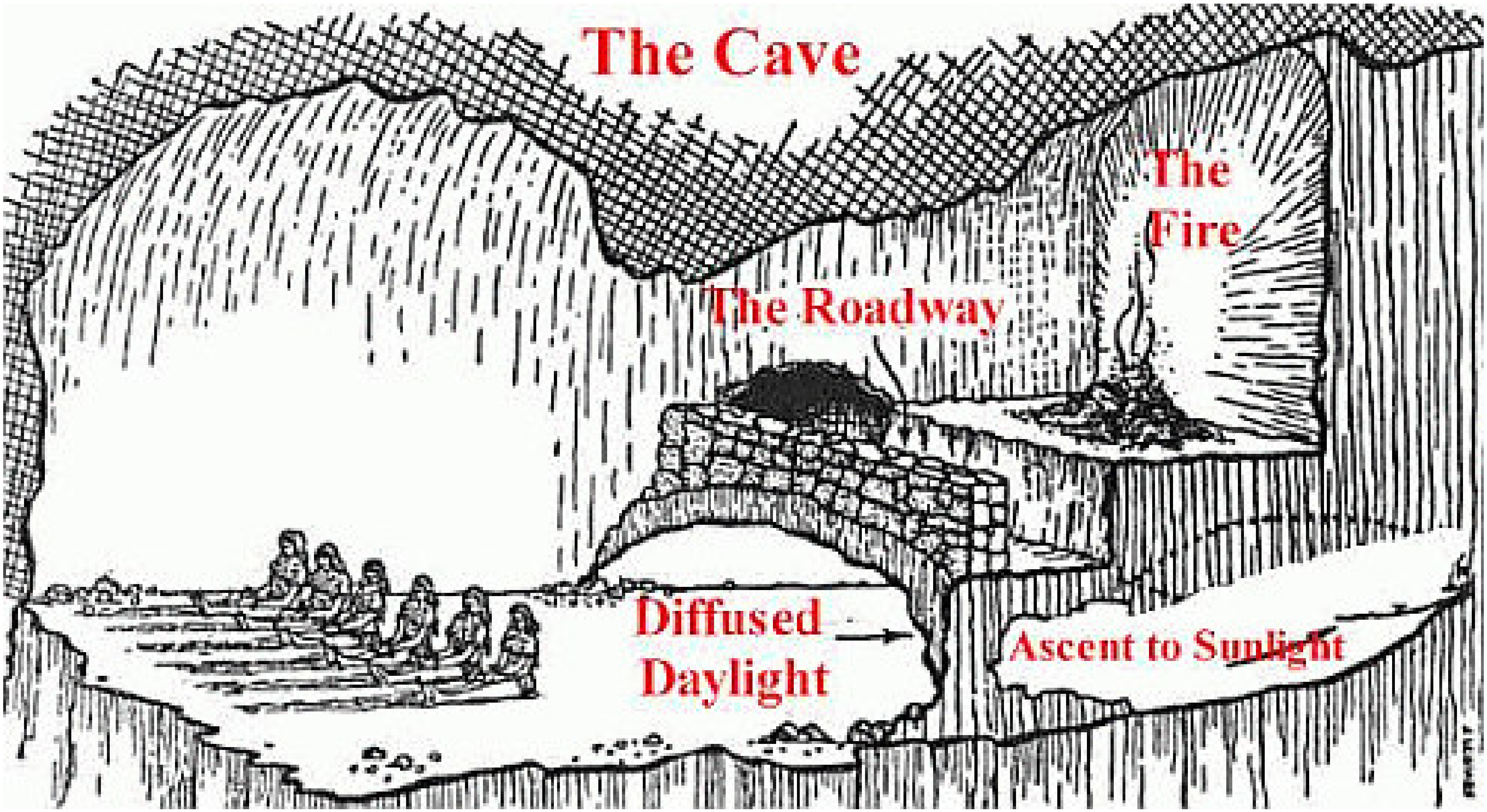
- In Plato's fictional dialogue, Socrates begins by describing a scenario in which **what people take to be real would in fact be an illusion.**
- He asks Glaucon to **imagine a cave inhabited by prisoners who have been chained and held immobile since childhood: not only are their legs (but not arms) held in place, but their necks are also fixed, so they are compelled to gaze at a wall in front of them.**

Inside the cave
[from wikipedia]

- Behind the prisoners is **an enormous fire, and between the fire and the prisoners is a raised walkway, along which people walk carrying things on their heads "including figures of men and animals made of wood, stone and other materials".**

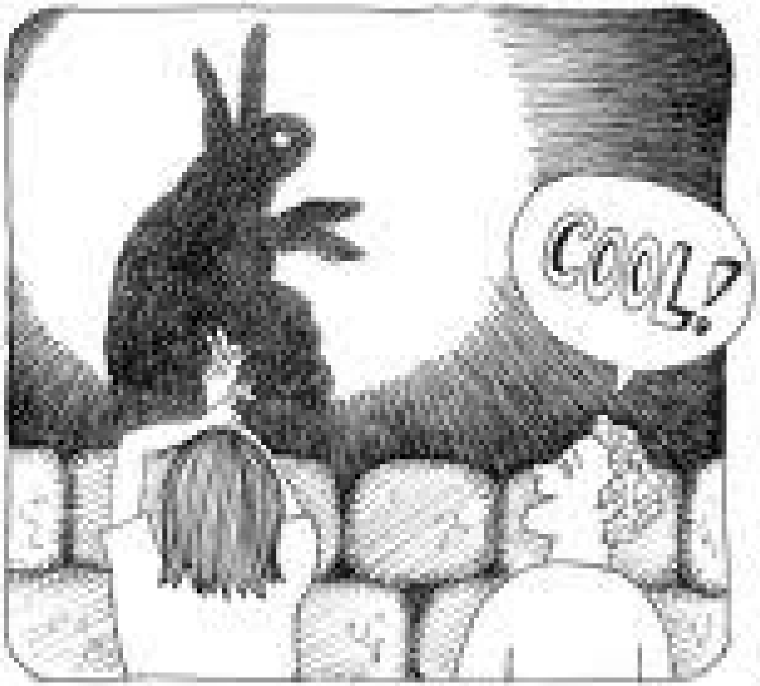
Inside the cave
[from wikipedia]

- **The prisoners** cannot see the raised walkway or the people walking, but they **watch the shadows cast by the men, not knowing they are shadows**. There are also echoes off the wall from the noise produced from the walkway.


The Cave
The Fire
The Roadway
Diffused
Daylight
Ascent to Sunlight

Inside the cave [from wikipedia]

- Socrates suggests the prisoners would take the shadows to be real things and the echoes to be real sounds created by the shadows, not just reflections of reality, since they are all they had ever seen or heard.


COOL!


THIS ONE REALLY GETS THEM GOIN'!
WOW!

Inside the cave [from wikipedia]

- They would praise as **clever**, whoever could **best guess which shadow would come next**, as **someone who understood the nature of the world**, and the whole of their society would depend on the shadows on the wall.
- http://en.wikipedia.org/wiki/Allegory_of_the_Cave

The Cave Problem

- What is the reality that is transformed by projection into the images that are experienced by those with limited vision?
- Philosophical questions:
  - What are real objects?
  - How do humans (with limited capacities) come to know the truth about real objects

Projections: Geometry

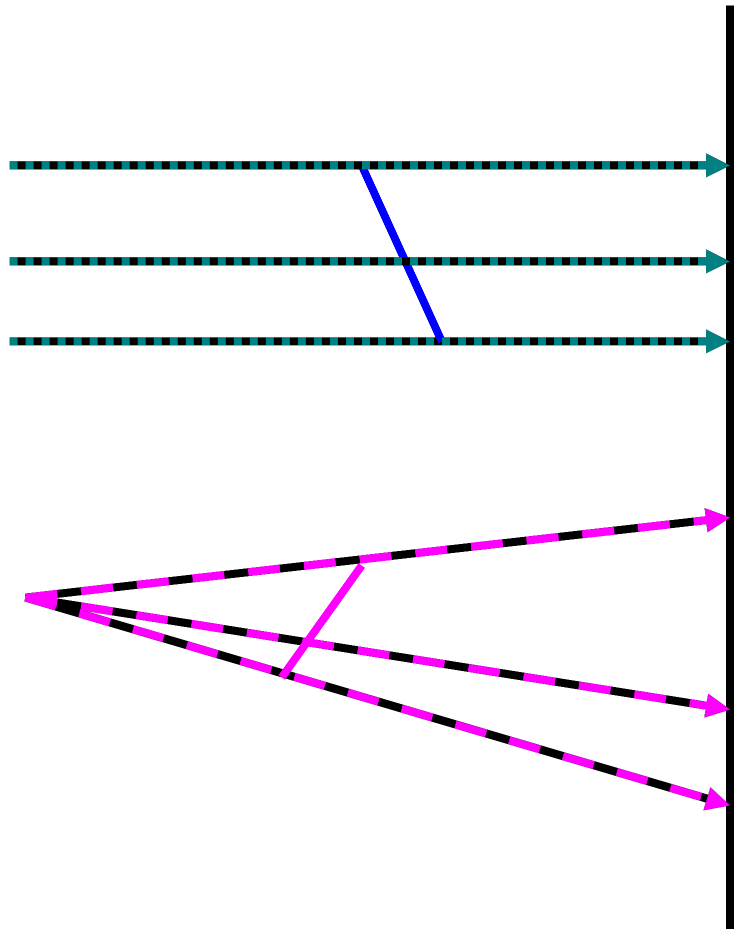
- Projections play an important role in geometry.
- There is even a special part of geometry called projective geometry- broken into two parts:
  (i) synthetic (axioms) and
  (ii) algebraic (coordinates and equations)

Projections in Geometry

- Examples: Central Projection and parallel projection
  - Points and Lines project to points and lines.
  - Triangles project to "triangles."
  - Circles can project to an ellipse, parabola, hyperbola or line: conics
  - Points of contact project to points of contact: tangent properties

Geometric Projections

- Parallel
- Central

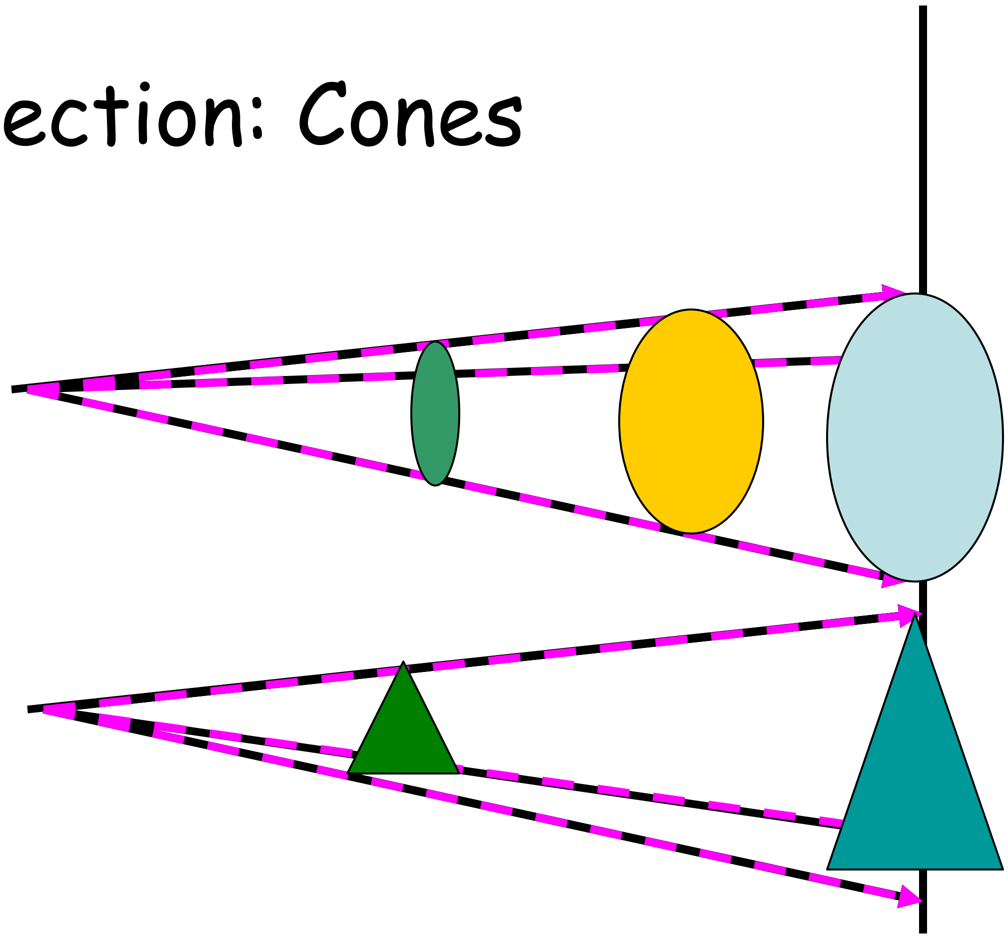


Geometric Projections

Central Projection: Cones

Circle

Triangle



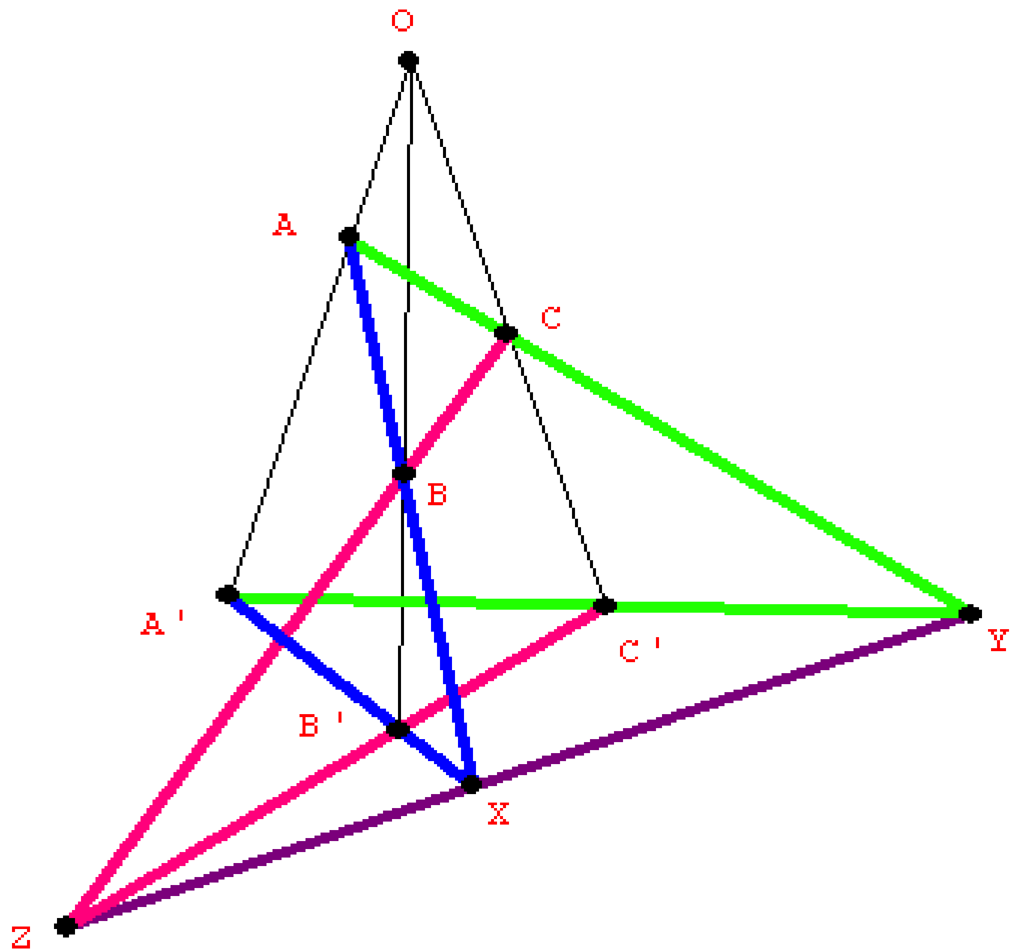
Geometric Projections

- Physical models for projections:
  - Line
  - Triangle
  - Tetrahedron
  - Cube
  - “Hypercube”

The Cave Problem

- What is the reality that was transformed by projection into the images that are experienced?
- Examples:
  - General scalene triangle from an Equilateral triangle… or vice versa?
  - Ellipses from circles… or vice versa?

Desargues' Theorem

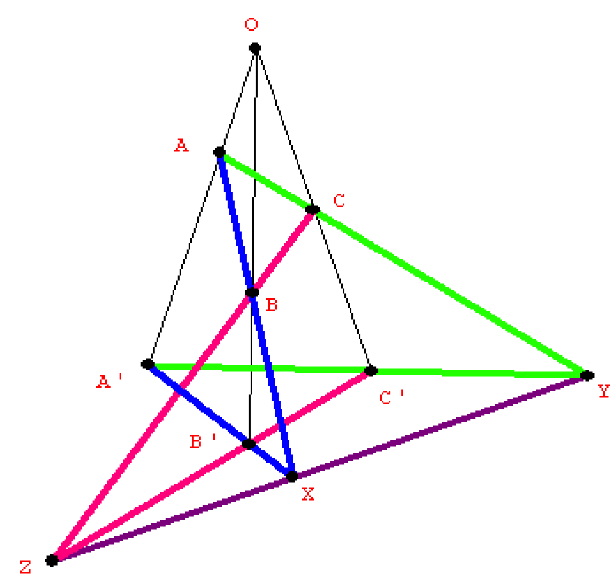


Desargues' Theorem

- Suppose triangle ABC is projected onto triangle A'B'C' from the central point O,
- The line AB meets the line A'B' at X;
- The line AC meets the line A'C' at Y;
- The line BC meets the line B'C' at Z.
- **Then** the points X,Y, and Z are on the same line.

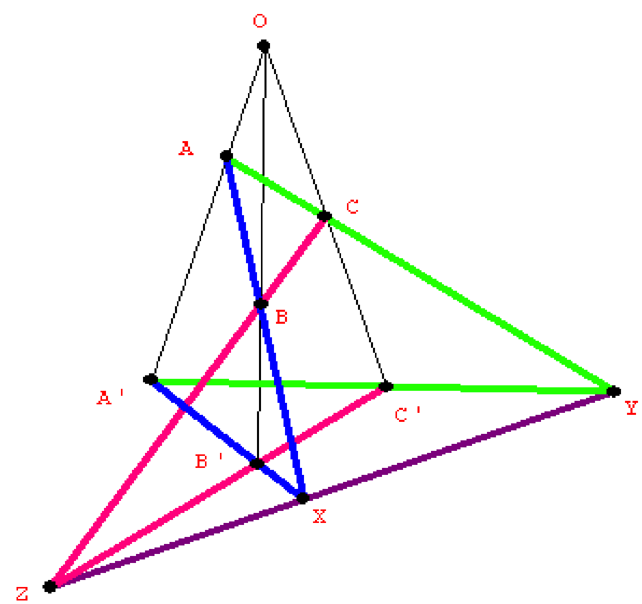
Desargues' Theorem

- If two non co-planar triangles ABC and $A'B'C'$ are perspectively related by the center O, then the points of intersection $X=AB\cap A'B'$; $Y=AC\cap A'C'$; and $Z=BC\cap B'C'$ all lie on the same line.



Desargues' Theorem

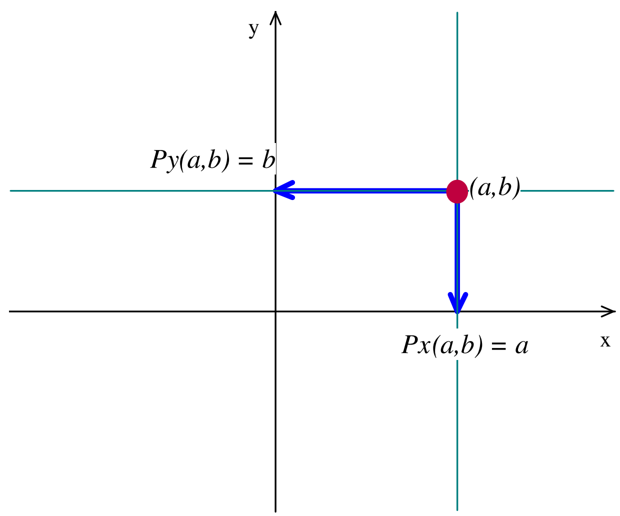
- If two co-planar triangles *ABC* and *A'B'C'* are perspectively related by the center O, then the points of intersection X = *AB*∩*A'B'*; Y = *AC*∩*A'C'*; and Z = BC∩B'*C'* a



Projection in Analytic coordinate geometry

Coordinate projection in the plane:

- Vertical projection:(a,b) → (a,0)
  - Px(a,b) = (a,0) " = a" (orthogonal to X axis)
- Horizontal projection: (a,b) → (0,b)
  - Py(a,b) → (0,b) " = b" (parallel to X-axis)



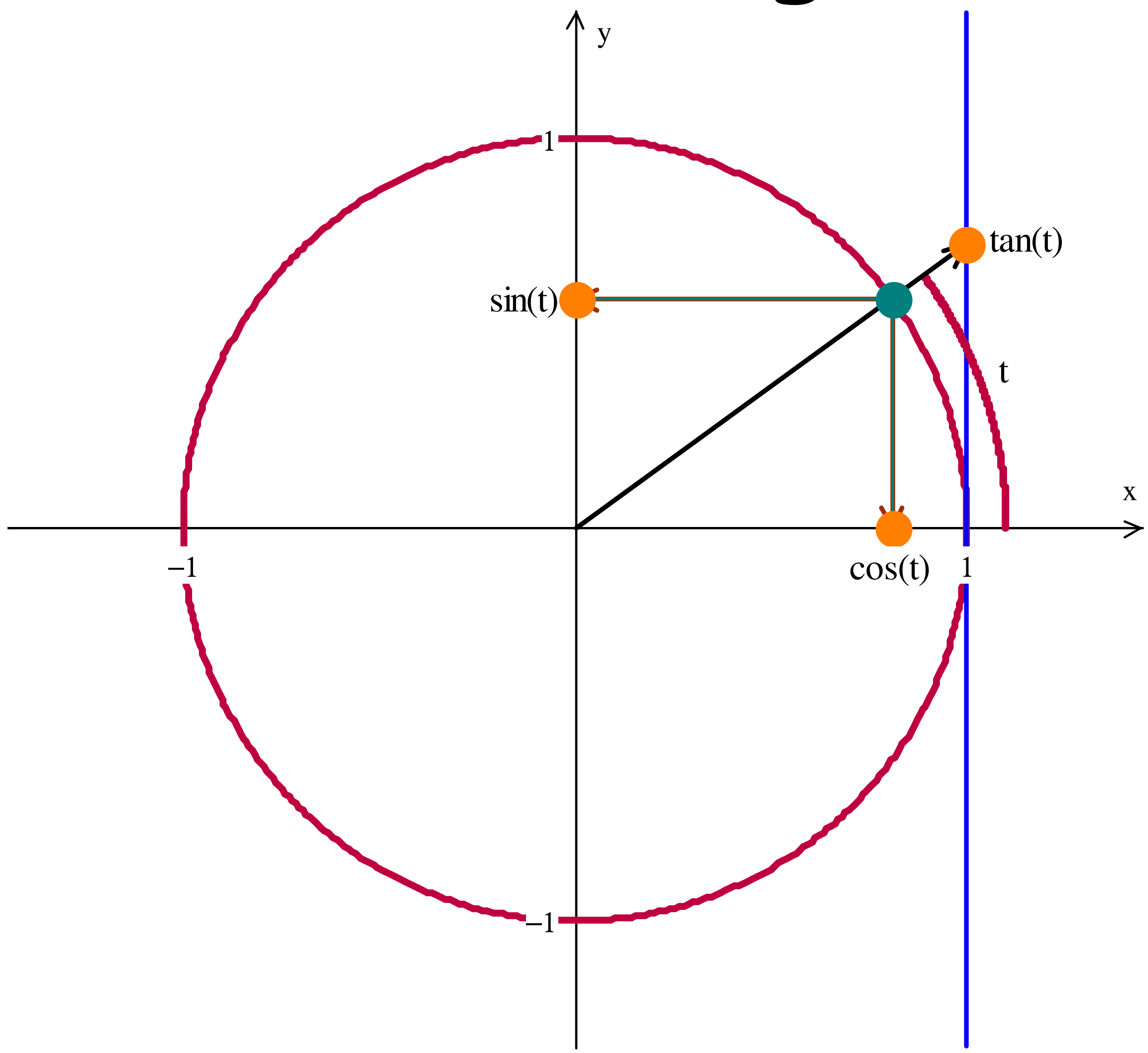
Projection in Analytic coordinate geometry

Coordinate projection in the plane: What is causing the shadow?

Simple Example of "**Unification**"

- Example 1
  - Suppose Px(a,b) = 5 and Py(a,b) = 3.
  - Conclusion: (a,b)=(5,3).
- Example 2
  - Suppose Px(a,b) = -2 and Py(a,b) = 7.
  - Conclusion: (a,b)=(-2,7).
- **Unification:**
  - Suppose Px(a,b) = r and Py(a,b) = s.
  - Conclusion: (a,b)=(r,s).

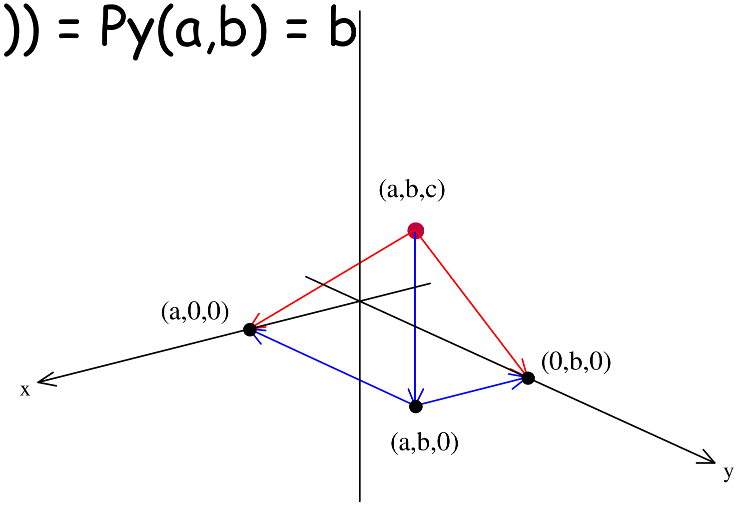
Projection and Trigonometry

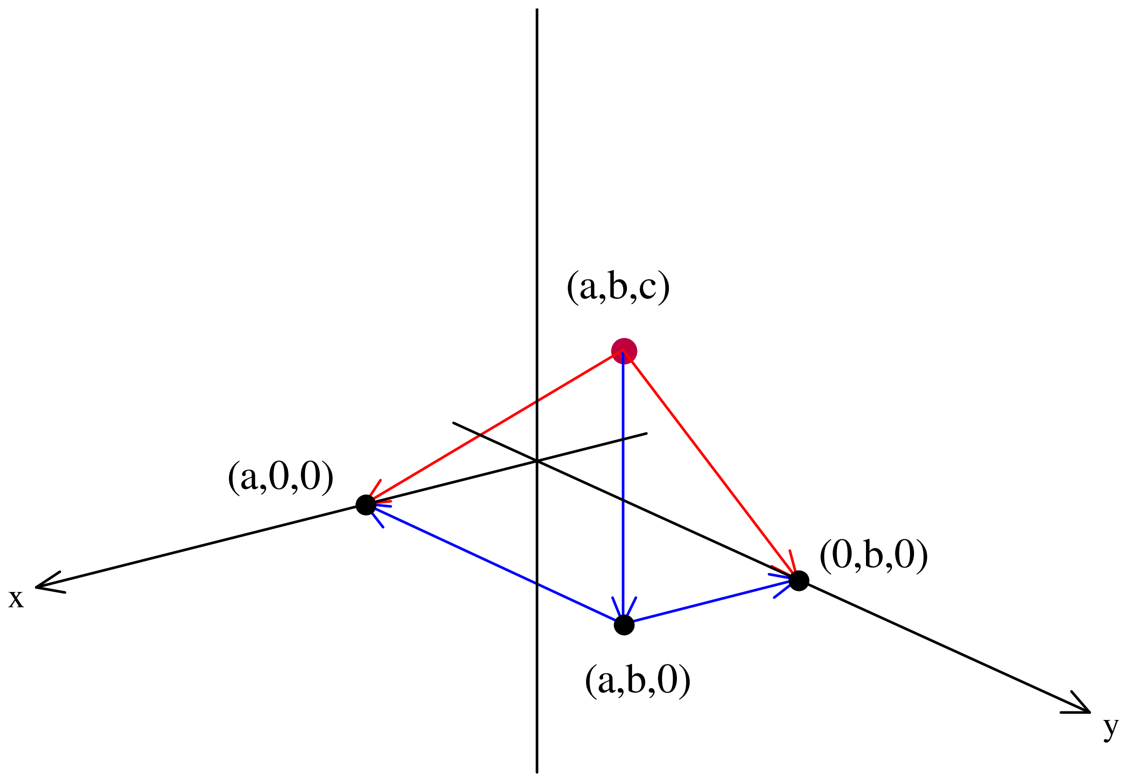


Projection in Spatial Coordinate Geometry

Coordinate projection in the space:
What is causing the shadow?

- Vertical projection: (a,b,c) → (a,b,0)
  - Pv(a,b,c) = (a,b,0) " = (a,b)" (orthogonal to the XY plane)
  - X projection: (a,b,c) → (a,0,0) (orthogonal to the X axis)
  - Px(a,b,c) = Px(Pv(a,b,c)) = Px(a,b) = a
  - Y projection: (a,b,c) → (0,b,0) (orthogonal to the Y axis)
  - Py(a,b,c) = Py(Pv(a,b,c)) = Py(a,b) = b



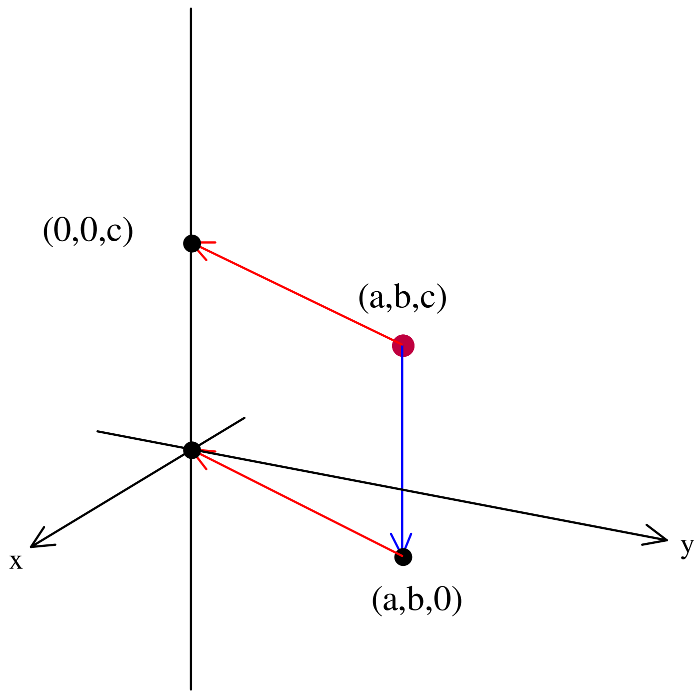


(a,b,c)
(a,0,0)
(0,b,0)
(a,b,0)
x
y

Projection in Analytic coordinate geometry

Coordinate projection in the space:
What is causing the shadow?

- Horizontal Z projection: (a,b,c) → (0,0,c)
  - Pz(a,b,c) → (0,0,c) = c (orthogonal to the Z axis)



Projection in Analytic coordinate geometry

Coordinate projection in the space:
What is causing the shadow?

Example of "**Unification**"

- Example 1
  - Suppose Px(a,b,c) = 5; Py(a,b,c) = 3; Pz(a,b,c) = 7
  - Conclusion: (a,b,c)=(5,3,7).
- Example 2
  - Suppose Px(a,b,c) =-2; Py(a,b,c) = 7; Pz(a,b,c) = 3
  - Conclusion: (a,b,c)=(-2,7,3).
- **Unification:**
  - Suppose Px(a,b,c) = r; Py(a,b,c) = s; Pz(a,b,c) = t
  - Conclusion: (a,b,c)=(r,s,t).

Projection in n-Space coordinate geometry

<u>Generalization</u>

Coordinate projection in the n-space:

What is causing the shadow?

- X_k projection for k = 1,2,..,n :

  $(a_1,a_2,\ldots, a_n) \rightarrow (0,0,\ldots,a_k,\ldots,0)$

  - $P_k (a_1,a_2,\ldots, a_n) = a_k$

- **Generalization of "result'**:

  - Suppose $P_k (a_1,a_2,\ldots, a_n) = r_k$ for k = 1,2,..,n ;
    Conclusion: $(a_1,a_2,\ldots, a_n) = (r_1,r_2,\ldots, r_n)$.

Projection of vector: Inner Product in R^n

- R^2 **<x,y>•<r,s> = xr + ys** $\mathbf{e}_1 = \langle 1,0\rangle$ $\mathbf{e}_2 = \langle 0,1\rangle$
- **If** $\mathbf{v}\cdot\mathbf{e}_1 = c_1$, $\mathbf{v}\cdot\mathbf{e}_2 = c_2$
  then $\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$
- R^3 **<x,y,z>•<r,s,t> = xr + ys + zt**

  $\mathbf{e}_1 = \langle 1,0,0\rangle$ $\mathbf{e}_2 = \langle 0,1,0\rangle$ $\mathbf{e}_3 = \langle 0,0,1\rangle$
- **If** $\mathbf{v}\cdot\mathbf{e}_1 = c_1$, $\mathbf{v}\cdot\mathbf{e}_2 = c_2$, $\mathbf{v}\cdot\mathbf{e}_2 = c_2$
  then $\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$
- **Generalization:**
- R^n : $\mathbf{e}_1 = \langle 1,0,0,\ldots,0\rangle$ $\mathbf{e}_2 = \langle 0,1,0,\ldots,0\rangle$ $\cdots$ $\mathbf{e}_n = \langle 0,0,0,\ldots,1\rangle$
- **If** $\mathbf{v}\cdot\mathbf{e}_1 = c_1$, $\mathbf{v}\cdot\mathbf{e}_2 = c_2, \ldots, \mathbf{v}\cdot\mathbf{e}_n = c_n$
  then $\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 + \ldots + c_n\mathbf{e}_n$

Projection in A Finite Dimensional Inner Product Space

- **V: A finite dimensional (real) vector space of dimension n with an "inner product".**
- **The inner product of vectors v and w denoted by v•w.**
- **Suppose we have a set of vectors**

  $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ where

  $\mathbf{v}_i \bullet \mathbf{v}_k = 0$ when $\mathbf{i} \neq \mathbf{k}$ **and** $\mathbf{v}_i \bullet \mathbf{v}_i = 1$.

- **Abstraction: If** $\mathbf{v} \bullet \mathbf{v}_1 = c_1$, $\mathbf{v} \bullet \mathbf{v}_2 = c_2, \ldots, \mathbf{v} \bullet \mathbf{v}_n = c_n$ then $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \ldots + c_n\mathbf{v}_n$

Decimal "Projections"

- The "mth decimal digit" as a projection of real numbers. Express a as a decimal (terminated if possible) $a = \Sigma a_k 10^{-k}$

  $k = -n, \ldots 0, 1, 2, \ldots$ with

  a_k an integer $0 \le a_k \le 9$.

  For m any integer, let $P_m(a) = a_m$
- Example: $P_3(\text{sqrt}(2)) = 4$; $P_2(\pi) = 4$
- Example: $P_k(x) = 0$ for $k < 0$, $P_0(x) = 3$; $P_1(x) = 1$; $P_2(x) = 4$. What is x?

Rational Numbers and Projections

- **Proposition**
  "rational numbers are repeating decimals":
- *A* real number *x* is a rational number **<u>if and only if</u>** there exist integers N and r,
  where if n >N, then $P_{n+r}(x)=P_n(x)$.
  <u>Proof organization/outline: (Polya's Steps)</u>
- To understand the problem, examine long division for 3/5 and 3/7-
  The plan: unify these examples.
- To understand the problem, convert .35 and .35353535…. to ratios of natural numbers –
  The plan: Unify these examples.

Number (function) projections

- Projection determined by a specific linear function P evaluation.

  $P(x) = 6x – 4$ "Solve an equation":

- Example 1: $P(x) = 8$: What is x?

  Solution: $6x - 4 = 8$; $6x = 12$; $x = 2$....CHECK!

- Example 2: $P(x) = 5$: what is x?

  Solution: $6x -4 = 5$; $6x = 9$; $x = 9/6 = 3/2$....CHECK!

- **Unification**

  P(x) = r: What is x?

  Solution: $6x -4 = r$; $6x = r +4$; **$x = (r+4)/6$**....CHECK!

Number (function) projections:

- "Projection determined by general linear function P. $P(x) = mx+b \quad m \neq 0$. "Solve an equation":
- Example 1: $P(x) = 8$: what is x?

  Solution: $mx+b = 8$; $mx = 8-b$; $x = (8-b)/m$....CHECK!
- Example 2: $P(x) = 5$: what is x?

  Solution: $mx +b = 5$; $mx = 5-b$; $x = (5-b)/m$....CHECK!
- **Unification/generalization**

$P(x) = r$: what is x?

Solution: $mx+b = r$; $mx = r - b$;

$x = (r-b)/m$....CHECK!

"Projection" and Linear Transformation
Evaluation as Projection!

- "Projection" Information with a **linear transformation**: $T: R^n \rightarrow R^n$
- $T: R^2 \rightarrow R^2$ $\mathbf{e_1} = \langle 1,0 \rangle$ $\mathbf{e_2} = \langle 0,1 \rangle$
  - If $T(\mathbf{e_1})=\mathbf{v_1}$, $T(\mathbf{e_2})=\mathbf{v_2}$ then $T(a_1,a_2) = a_1\mathbf{v_1} + a_2\mathbf{v_2}$
  - Proof: $T(\langle a_1,a_2 \rangle) = T(a_1\mathbf{e_1} + a_2\mathbf{e_2})$
    $= a_1\mathbf{T(e_1)} + a_2\mathbf{T(e_2)}$
    $= a_1\mathbf{v_1} + a_2\mathbf{v_2}$
- $T: R^3 \rightarrow R^3$
  $\mathbf{e_1} = \langle 1,0,0 \rangle$ $\mathbf{e_2} = \langle 0,1,0 \rangle$ $\mathbf{e_3} = \langle 0,0,1 \rangle$
  If $T(\mathbf{e_1})=\mathbf{v_1}$, $T(\mathbf{e_2})=\mathbf{v_2}$, $T(\mathbf{e_3})=\mathbf{v_3}$ then
  $T(\langle a_1,a_2,a_3 \rangle) = a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3}$

"Projection" in Linear Transformation Generalization

- "Projection" Information with a linear transformation: T: $R^n \rightarrow R^n$

$\mathbf{e_1}$= <1,0,0,…,0> $\mathbf{e_2}$ =<0,1,0,…,0> … $\mathbf{e_n}$= <0,0,0,…,1>

- If $T(e_1) = \mathbf{v_1}$, $T(e_2) = \mathbf{v_2}$,…, $T(e_n) = \mathbf{v_n}$ then

  $T(\langle a_1,a_2,\ldots, a_n\rangle) = a_1\mathbf{v_1} + a_2\mathbf{v_2} +\ldots+ a_n\mathbf{v_n}$

Projection of complex numbers

- $P_n(z) = z^n$ (nth Power function)
- $T_n(z) = \Sigma \frac{z^k}{k!}; k = 0,1, \dots, n$ (a polynomial)
- $P\ (z) = \Sigma \frac{z^k}{k!}; k = 0,1, \dots = \text{“} e^z \text{”}$
- If $P(z) = 1$ then $z = 0.$
  If $P(z) = e$ then $z = 1$
- If $P(z) = L$ then $z = \ln(L).$

Projection of complex numbers

- $P(z) = \Sigma \frac{z^k}{k!}$; $k = 0,1, \ldots =$ " e^z"
- $P(bi) = \cos(b) + isin(b)$ (from power series)
  $= (\cos(b), \sin(b))$ in Complex Plane Geometry
- $P(a + bi) = e^a\left(\cos(b) + isin(b)\right)$
  $= (e^a \cos(b), e^a \sin(b))$

Projection of complex numbers

- $P(bi) = \cos(b) + isin(b)$
  $= (\cos(b), \sin(b))$ in Complex Plane
  Geometry
- $P(a + bi) = e^{a}(\cos(b) + isin(b))$
  $= (e^{a}cos(b), e^{a}\sin(b))$

Suppose $z = a + bi$

- Example 1: If $P(z) = 1$, what is z?

Solution: $e^{a} = 1$, so $a = 0$;
$1 = \cos(b)$, so $b = 2\pi k$ where k is any integer.
$[\sin(b) = 0$ is consistent]

Projection of complex numbers

- $P(bi) = \cos(b) + isin(b)$
  $= (\cos(b), \sin(b))$ in Complex Plane Geometry
- $P(a+bi) = e^a(\cos(b) + isin(b))$
  $= (e^a cos(b), e^a \sin(b))$

Suppose $z = a + bi$

- Example 2: $P(z) = -1$ what is z?

Solution: $e^a = 1$, so $a = 0$;
$-1 = \cos(b)$, so $b = \pi(2k+1)$ where k is any integer. [$\sin(b) = 0$ is consistent]

In particular when k = 0, z = 0 + πi and

$e^{\pi i} = -1$ and $\ln(-1) = \pi i$

or $\ln(-1) = (2k+1)\pi i$

Number (modular) projections

- Projection determined by specific result from division.
  Example: $\mathbf{P_5(x) = R}$ where $x = Q5 + R$; **Q is an integer and** $\mathbf{0 \leq R < 5}$.
  - Example 1: $\mathbf{P_5}$ $(x) = 0$: what is x?
    Multiple Solutions: x = 0, 5, 10, … , Q5 where Q is an integer.
    ….CHECK
  - Example 2: $\mathbf{P_5}(x) = 2$: what is x?
    Multiple Solutions: x = 2, 7, 12, …, 2 + Q5 where Q is an integer.
    ….CHECK!
  - **<u>Unification</u>**

    $P_5(x) = r$: what is x?

    Solution: $x = r + Q5$ where Q is an integer.
    ….CHECK!

Number (modular) projections:

- Projection given by general result from division.
  $P_n(x) = R$ where $x = Qn + R$; **Q is an integer**
  and $\mathbf{0 \leq R < n}$.
- Example 1: $P_n(x) = 0$: what is x?
  Multiple Solutions: $x = Qn$ where Q is an integer. ….CHECK
- Example 2: $P_n(x) = 2$: what is x?
  Multiple Solutions: $x = 2 + Qn$ where Q is an integer.
  ….CHECK!

Unification/Generalization

$P_n(x) = r$: what is x?

Solution: $x = r + Qn$ where Q is an integer.
….CHECK!

Projection/Evaluation of a curve

- Projection of curve: Is this the graph of a polynomial function? [trigonometric? exponential?]
- One value: constant
- Two values: Linear polynomial
  [A sin(Bx),A e^(Bx)]
- 3 values: quadratic polynomial
  [Asin(Bx +C), Ae^(Bx)+C]
- N+1 values: polynomial of degree N
  Lagrange Interpolation

Lagrange Interpolation

- N+1 values: polynomial of degree N
  $P(a_1) = b_1$; $P(a_2) = b_2$; $P(a_3) = b_3$;… ; $P(a_{n+1}) = b_{n+1}$

$P(x) = b_{n+1} (x-a_1) (x-a_2)…(x-a_n)/[(a_{n+1}-a_1) (a_{n+1}-a_2)…(a_{n+1}-a_n)]$

$+ b_n (x-a_1) (x-a_2)…(x-a_{n-1}) (x-a_{n+1})/[(a_n-a_1) (a_{n+1}-a_2)… (a_n-a_{n-1})(a_n-a_{n+1})] +$

…

$+ b_1 (x-a_2)… (x-a_{n+1})/[(a_1-a_2)…(a_1-a_n) (a_1-a_{n+1})]$

Projection of curve: polynomial? (derivatives)

- One value f(a): constant $P(x) = f(a)$
- Two values f(a), f'(a):
  Linear $P(x)=f(a) + f'(a)(x-a)$
- 3 values f(a), f'(a), f''(a):
  quadratic $P(x) = f(a) + f'(a)(x-a) + f''(a)/2\ (x-a)^2$
- N+1 values f(a), f'(a), f''(a) … $f^N(a)$: poly of degree N
  - $P(x)=f(a)+f'(a)(x-a)+f''(a)/2\ (x-a)^2+\ldots+f^N(a)/(N!)\ (x-a)^N$
- Infinite values: f(a), f'(a), f''(a) …
  "Taylor Series" for x
  - $P(x)=f(a)+f'(a)(x-a)+f''(a)/2\ (x-a)^2+\ldots+f^N(a)/(N!)\ (x-a)^N +\ldots$

Taylor Series for f

- Infinite values: f(a), f'(a), f''(a) … f is called (real) **analytic** if f(x) = Taylor Series for f.
- Warning Example:

$f(x) = \exp(-1/x^2)$ for $x \neq 0$ and $f(0) = 0$.

Then it can be shown that the (MacLaurin) Taylor series for f at a=0 is the 0 function.

So f is not analytic, and

knowing all the derivatives of f at a single point is not enough to determine the function for real valued function of real numbers.

[For complex functions things are better!]

Projection of a Periodic curve

- One value f(a): constant
- Two values: $A_0 + A_1 \sin(x)$
- 3 values: $A_0 + A_1 \sin(x) + B_1\cos(x)$
- 2N+1 values poly of degree: $A_0 + A_1 \sin(x) + B_1\cos(x) + ... + A_N \sin(Nx) + B_N\cos(Nx)$
  Infinite… Series for
  $A_0 + A_1 \sin(x) + B_1\cos(x) + \ldots + A_N \sin(Nx) + B_N\cos(Nx) + \ldots$

Invariance: "fixed points"

- f: [0,1]->[0,1] . Is there an x with f(x) = x?
  - If f is continuous, yes!
    PROOF: ?

Unify

- f: [a,b]->[a,b] . Is there an x with f(x) = x?
  - If f is continuous, yes!
    PROOF:?

Generalize

- $f\colon [0,1]^n \to [0,1]^n$. Is there an x with f(x) = x?

  - If f is continuous, yes!

Invariance: "fixed points"

- T: V ->V T a linear operator: is 1 an eigenvalue? Does there exist v in V, where T(v)=v?

- If T is not the constant 0 operator and TT=T [T is called a projection operator] then yes!
  Proof: T(T(v))=T(v) for all v in V. But T is not 0, so there is some w* where T(w*) is not the 0 vector. Let v*= T(w*). Then T(v*) = T(T(w*))=T(w*)=v*.

- **Abstract:**
  $T\colon M \to M$ Is there an x in M with T(x) = x?
- If TT=T, then yes!

- In Metric space, M: F is a **contraction** if
  d(F(x),F(y)) = k d(x,y) where 0<k<1.
- Theorem: If M is a complete metric space and F is a contraction then F has a fixed point.

Projections from higher dimensions!

Hypercubes,
from dimension 0 up to dimension 6

Thanks!

Questions

The End.

Context

It is important to understand the role that **context** plays with these three uses in mathematical thought.

Comment: Context was a key element of Bertrand Russell's analysis of the use of language in "On Denoting".
It is an important element in understanding interpretations in foundations and model theory.

Context Connections

- **U**nification: A single context.
- **G**eneralization: Distinct but connected, related contexts.
- **A**bstraction: Broad context definition with a structural characterization that allows discourse with limited specificity.

Approach to Proof and Problem Solving

- Balanced Focus
  - **Polya steps in light of Context and organization**

1. <u>**Understand the context**</u> of the statement(s).
   - Is this a **unification,**
   - **generalization, or**
   - **abstraction?**
2. **R**ecognize how to connect understanding of instances and the context to **develop a plan** with an appropriate <u>**organization**</u>.
3. **Execute the plan** at the appropriate level of discourse and with a clear <u>**organization**</u>**.**
4. **Reflect on the role of context and how that effected the plan and its execution.**

Examples of Unification

Addition of vectors: Define (a,c) +(b,d) to be the vector pair (a+b, c+d).
Fact: (3,4)+(0,0) = (3,4)
Fact: (-4,7)+(0,0) = (-4,7)
Fact: (8,-5)+(0,0) = (8,-5)
Unification: For any a and c, (a,c)+(0,0) = (a,c)

Fact: (3,4)+(-3,-4) = (0,0)
Fact: (-4,7)+(4,-7) = (0,0)
Fact: (8,-5)+(-8,5) = (0,0)
Unification: For any (a,b), (a,b)+(-a,-b) = (0,0)

Inner Product of Vectors: Define (a,c) ◦ (b,d) to be the number $a \cdot b + c \cdot d$.
Fact: (3,4)◦(-4,3) = 0
Fact: (-4,7)◦(-7,-4) = 0
Fact: (8,-5)◦(5,8) = 0
Unification: For any (a,b), (a,b) ◦ (-b,a) = 0

Examples of Generalization

Generalization of Addition:
Define (a1,a2, …, an) +(b1,b2,…, bn) to be the n-tuple (a1+b1, a2+b2, …, an+bn).

Unification: For any a and b, (a,b)+(0,0) = (a,b)
Generalization: For any a1,a2, …, an, (a1,a2, …, an) +(0,0, …, 0) = (a1,a2, …, an)

Unification: For any (a,b), (a,b)+(-a,-b) = (0,0)
Generalization: For any (a1,a2, …, an) ,
(a1,a2, …, an) + (-a1,-a2, …, -an) = (0,0, … , 0)

Generalization of Inner Product of Vectors for n-tuples:
Define (a1,a2, …, an) ◦(b1,b2,…, bn) to be the number a1b1+a2b2+… + anbn.

Unification: For any (a,b), **(a,b) ◦ (-b,a) = 0**
Generalization: For any (a1, a2,…,an), **(a1, a2,…,an) ◦ (-a2, a1, 0, … 0) = 0**

Examples of Abstraction

Abstraction of Addition: (Group, Ring, Field, Vector Space,…)
An Addition is a (commutative) binary operation "+" on set S for x and y in S to be the element of S, "x+y" with the property that x+y = y+x for any x and y in S.

Unification: For any a and b, (a,b)+(0,0) = (a,b)
Generalization: For any a1,a2, …, an, (a1,a2, …, an) +(0,0, …, 0) = (a1,a2, …, an)
Abstraction: "Additive identity" z in S is an additive identity: for any x in S, x + z = x.

Unification: For any (a,b), (a,b)+(-a,-b) = (0,0)
Generalization: For any (a1,a2, …, an) , (a1,a2, …, an) + (-a1,-a2, …, -an) = (0,0, … , 0)
Abstraction: "Additive inverse" For any x in S there is an element y of S with x + y = z .

Generalization of Inner Product of Vectors:
We say an operation "◦ " on a real vector space V is an inner product if for any v and w in V, v ◦w is a real number and the following properties are true: ……..

Unification: For any (a,b), (a,b) ◦ (-b,a) = 0
Generalization: For any (a1, a2,…,an), (a1, a2,…,an) ◦ (-a2, a1, 0, … 0) = 0
Abstraction: Suppose dim(V) >1, then for any v in V, there is a nonzero vector w in V with v ◦w = 0.

Examples for Understanding the Context of a Statement.

1. For a and b real numbers, $a \neq 0$, there is a unique real x with ax=b.
2. For a real number, $a \neq 0$, and an n-tuple of real numbers (b1,b2,…,bn) there is a unique n-tuple of real numbers (x1,x2,…,xn) with
   a (x1,x2,…,xn) = (b1,b2,…,bn).
3. For a scalar, $a \neq 0$, in the field F and a vector **b** in a vector space **V** over the field F, there is a unique vector **x** in **V** with a**x** = **b**.

Unification

1. For a and b real numbers, $a \neq 0$, there is a unique real x with $ax=b$.

- A unification in the context of the algebra of real numbers.
- Continue understanding by examining some cases, such as $a = 5$ and $b = 12$.
- Plan based on experience with algebra of real numbers.

Generalization

2. For a real number, $a \neq 0$, and an n-tuple of real numbers (b1,b2,…,bn) there is a unique n-tuple of real numbers (x1,x2,…,xn) with

   a (x1,x2,…,xn) = (b1,b2,…,bn).

- A generalization in the context of the algebra of real n- dimensional vectors in R^n .
- Continue understanding by examining some cases, n = 2,3, etc.
- Plan based on aspects of examples that generalize to all specified contexts.

Abstraction

3. For a scalar, $a \neq 0$, in the field F and a vector **b** in a vector space **V** over the field F, there is a unique vector **x** in **V** with a**x** = **b**.

- An abstraction in the context of the algebra of vectors in a vector space over a field.
- Continue understanding by examine some specific included contexts.
- Plan based on aspects of examples that rely only on abstract qualities of the examples.

Ideas for Talk

- Projections in geometry
  - Shadows (Plato)
  - Projective Geometry: Properties preserved by projections: perspective, incidence, "linearity", "conics"
  - Proof by projection: Desargues' Theorem, Brianchon's Theorem

  - Function definitions: functions as projections –
    - f: [a,b] → [c,d] onto… 1:1!
    - $f : x \to x^2$ onto $[0, \infty)$ "2:1"
    - Wrapping function f: R → unit circle.
    - Trig functions as projections
  - Operators as projections: P: S (unknown) →T ("known")
    - Dn :C inf Functions → Dnf(0)
    - Dn :C inf Functions → Dnf
  - **Linear algebra: Operator P st PP=P.**
  - Coverings: f: X -> B onto and "continuous"
    - X=R, B = Circle
    - X = R2 B = torus
    - X= R2 B = 2 holed torus