"The Role of Philosophy in Proofs"

Martin Flashman
Humboldt State University
Arcata, CA
flashman@humboldt.edu
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Preparation Questions

- How many of you have taken or are currently taking the transition to proof course? **MATH 240: Introduction to Mathematical Thought**
- How many of you have been taught "logic" with truth tables for propositions and venn diagrams for quantification?
- How many of you have discussed some aspect of the philosophy of mathematics in your courses?

Polya's 4 Phases of Problem Solving

1. **Understand the problem.**
2. **See connections** to devise a plan.
3. Carry out the plan.
4. Look back. **Reflect on the process and results.**

Question: What role can philosophy play in problem solving and "proof".

Mathematics and Proof

Mathematical proof is not identical to a demonstration involving only truth tables and the syntax of quantification.

Material implication and quantification are used in mathematics because **in mathematics the concern is on contexts where conditional statements and quantification have significant meaning**.

Philosophical Questions

What is the nature of the objects of mathematical discourse ? Ontology.

What is the nature of mathematical structures ? More Ontology.

What is the nature of mathematical knowledge ? Epistemology.

Ontology (Wikipedia)

- The philosophical study of the nature of *being*, *becoming*, *existence*, or *reality*, as well as the basic categories of being and their relations.
- What entities exist or can be said to exist, and how such entities can be grouped, related within a hierarchy, and subdivided according to similarities and differences.

Epistemology (Wikipedia)

- The branch of philosophy concerned with the nature and scope of knowledge and is also referred to as "theory of knowledge".
- What is knowledge?
- How can knowledge be acquired?
- To what extent can knowledge pertinent to any given subject or entity be acquired.
- How does the nature of knowledge relate to connected notions such as truth, belief, and justification?

Ontological Commitment

- In philosophy a "theory is **ontologically committed** (o.c.) to an object only if that object occurs in *all* the ontologies of that theory."
- Is geometry o.c. to points?
- Is arithmetic o.c. to numbers?
- Is mathematics o.c. to sets?

The Examples (as time permits): Consider 3 of these 6 Statements and Proofs

- **Euclid Book I Proposition 1**
  - ***To construct an equilateral triangle on a given finite straight line.***
- **Euclid Book IX Proposition 20**
  - ***Prime numbers are more than any assigned multitude of prime numbers.***
- Pythagoras (?):
  - *The square root of 2 is not a rational number.*
- Cantor:
  - *The rational (or algebraic) numbers are equi-numerous with the natural numbers.*
- Cantor:
  - *The real numbers (or points on a line segment) are infinite but not equi-numerous with the natural numbers.*
- **Russell:**
  - ***R = { S: S is not an element of S } is not a set.***

Euclid Book I Proposition 1

To construct an equilateral triangle on a given finite straight line.

Proof: Given finite straight line AB.

With center A construct circle O with radius AB.

With center B construct circle O' with radius AB.

Construct Segment AC from A to C, the point of intersection of O and O'.

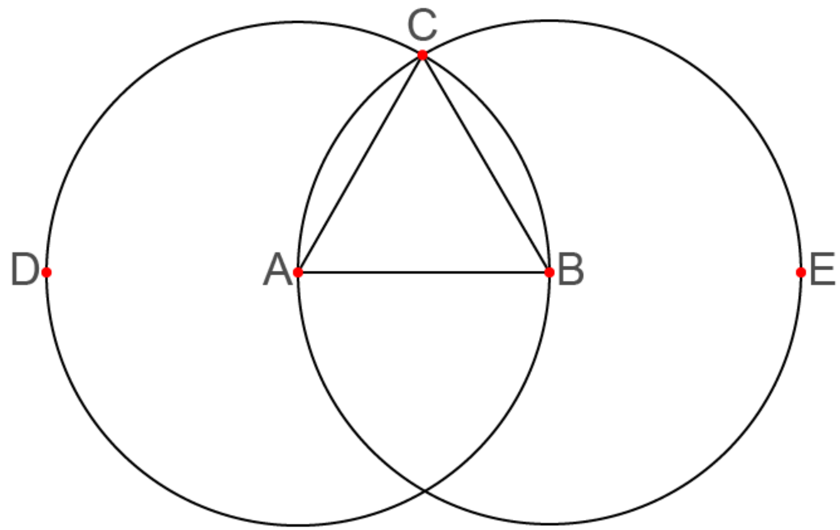
Construct Segment BC from B to C, the point of intersection of O and O'.

AC = AB.

BC = AB.

The triangle ABC is the desired equilateral triangle.

QEF.



Euclid Book I Proposition 1

To construct an equilateral triangle on a given finite straight line.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interests:

- Construction is existence. QEF vs QED
- Definitions based on primitives.
- Euclid Axioms built to model "reality".
- Hilbert approach to (formal) axioms for geometry.

Missing assumption:

The existence of point of intersection of circles.

- The power of counterexamples: Proofs and refutations (Lakatos)
- Alternative (models for) geometry :
  - Rational geometry.
  - Geometry without compass but with Playfair parallel postulate.

Euclid Book IX Proposition 20

Prime numbers are more than any assigned multitude of prime numbers.

Proof: (Modified from Euclid)

Suppose the primes comprise p1, p2, …, pn.

Let q = p1*p2* … *pn + 1.

Then q is not a prime.

But any number is either a prime or has a prime factor.

So one of the primes, p1, p2, … , pn, is a factor of q.

But that same prime is a factor of p1*p2* … *pn so it must be a factor of 1. This is absurd, so

The primes are more than any assigned multitude of prime numbers.

QED.

Euclid Book IX Proposition 20

Prime numbers are more than any assigned multitude of prime numbers.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interests:

- Existence without construction. QED (not QEF)
- Definitions and prior results in an information web. (Structures)
- Euclid definitions built to generalize multiple measurement contexts : length, area, volume.
- Euclid's actual "proof". Why is three enough?
- Peano axioms abstract structure and "implication" relationship. Russell- Whitehead build from abstract logic. Other foundations for numbers based on set measurement and equivalence relations.

Importance of consistency:

- Mathematics abhors contradiction within its structures.
- Indirect proof and construction depend on consistency.

Russell

R = { S: S is not an element of S } is not a set.

Proof:

Suppose R is a set.

Then the statement: "R is an element of R" is a proposition.

If R is an element of R, then by definition of R, R is not an element of R .

This is a contradiction, so R is not an element of R.

Now by definition of R, R is an element of R.

In summary, if R is a set there is a mathematical proposition that is a contradiction.

This is absurd. So R is not a set. Q.E.D.

Russell

R = { *S: S is not an element of S* } *is not a set.*

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Set theory needs some restrictions to be free from contradictions.

Can mathematics be analyzed by mathematics alone?

Is there a need for some philosophy to understand

mathematical objects ?

mathematical structures ?

mathematical knowledge ?

Logic is Not Epistemology

Should Philosophy Play a Larger Role in Understanding Proofs?

Answer:

?

The End

- Thanks-
- Questions

These slides will be posted at
http:// users.humboldt.edu/flashman