

# What can we learn from Newton's estimate of $\ln(2)$ ?

In Memory of Hank Tropp:  
Who gave me his copy of The Mathematical Works of  
Isaac Newton

Martin Flashman  
Department of Mathematics  
Humboldt State University  
[flashman@humboldt.edu](mailto:flashman@humboldt.edu)  
[users.humboldt.edu/flashman](http://users.humboldt.edu/flashman)

# Abstract

Newton made a very accurate estimate for the hyperbolic logarithm of 2 by combining understanding of properties of logarithms, the geometric series, and integration for polynomials.

The author will analyze Newton's approach and explore how this approach might be better understood by students by asking for an estimate of pi using the fact that

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}.$$

# Outline

- I. Analyze the approach used by Isaac Newton in his 16 decimal place estimate for the natural logarithm of 2 which appeared in “*The method of fluxions and infinite series*” [1671/1736]
- II. Examine briefly Newton’s estimate of  $\pi$  in the same publication to see how it follows a somewhat similar approach.
- III. Follow Newton’s approach from the logarithm more schematically by asking for an estimate of  $\pi$  using the fact

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}.$$

# Part I. Newton's computations of hyperbolic logarithms.

- In 1676 Newton wrote in a letter to Henry Oldenburg on some of his applications of series to estimating areas, in particular in estimating areas for the hyperbolic logarithm.
- This work was later clarified in *Of the Method of Fluxions and Infinite Series* which was published posthumously in 1737, ten years after Newton's death.

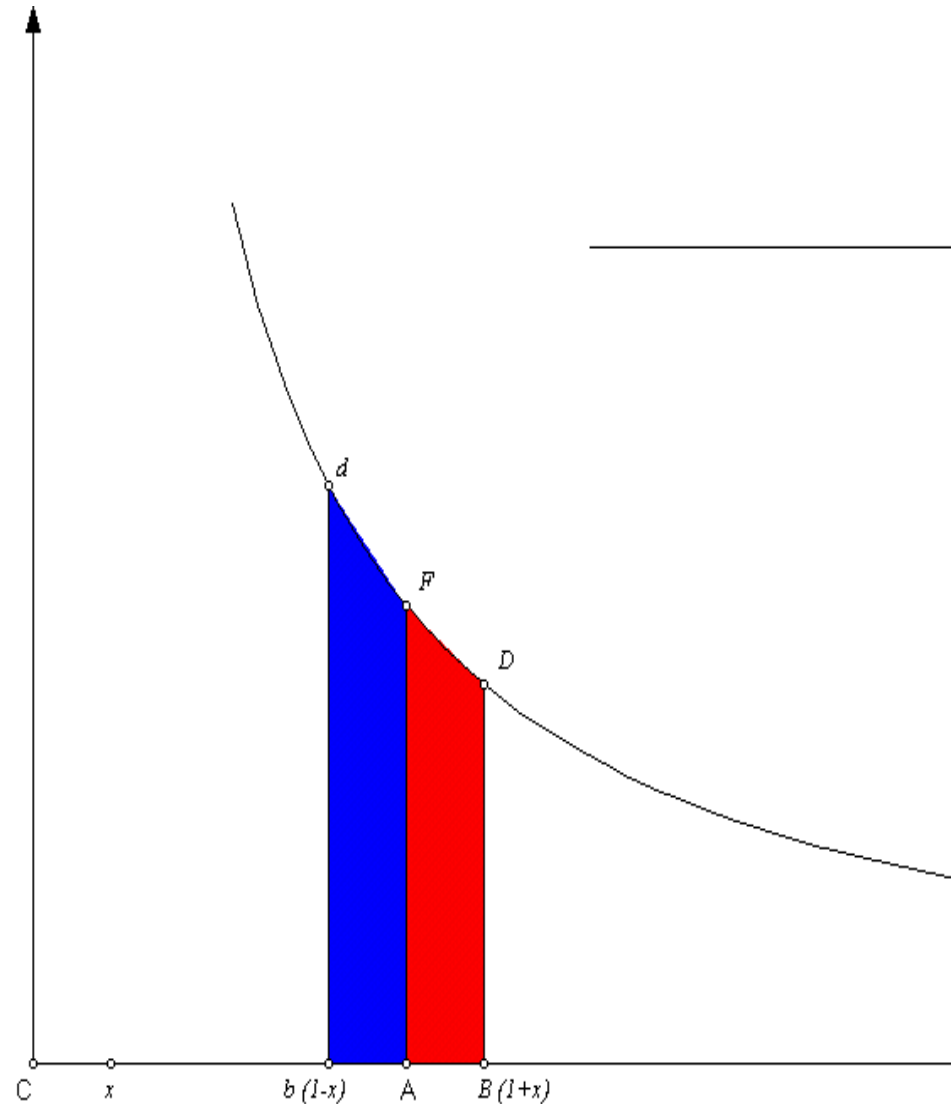


# Newton estimates the Hyperbolic Log

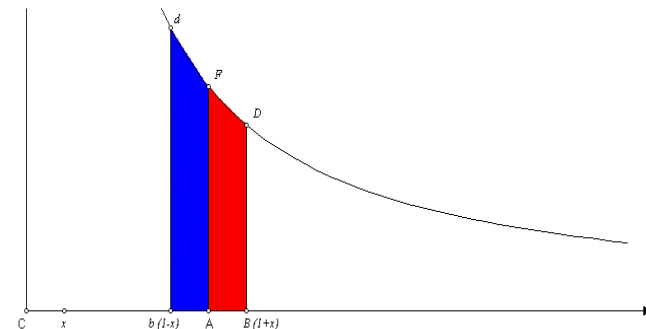
Newton considers symmetrically located points on the main axis,  $1 + x$  and  $1 - x$  with  $x > 0$  and their related reciprocals.

He then uses two integrals related to the geometric series to determine the related areas,

- (i) between the hyperbola and above the segment  $[1, 1 + x]$  (red) AFDB and
- (ii) between the hyperbola and above the segment  $[1 - x, 1]$  (blue) AFdb.



# The red and the blue.



$$\text{Area } AFDB = \int_0^k \frac{l}{l+x} dx = \int_0^k l - x + x^2 - x^3 + \dots = h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} \dots$$

$$\text{Area } AFdb = \int_0^k \frac{l}{l-x} dx = \int_0^k l + x + x^2 + \dots + x^k + \dots = h + \frac{h^2}{2} + \frac{h^3}{3} + \dots + \frac{h^k}{k} + \dots$$

These allow the estimation of the sum and difference of the two areas:

$$\text{Total area } bdDB = 2h + 2\frac{h^3}{3} + 2\frac{h^5}{5} + 2\frac{h^7}{7} + \dots$$

$$\text{Difference of areas } Ad - AD = h^2 + \frac{h^4}{2} + \frac{h^6}{3} + \frac{h^8}{4} + \dots$$

Now to find the Area of the two separate regions (and related logarithms) we take  $1/2$  of the difference of these results and  $1/2$  of the sum of these results.

- Newton uses the first eight terms with  $h = .1$  and  $h = .2$  to estimate *the hyperbolic (natural) logarithm* of

$0.9, 1.1, 0.8$  and  $1.2$

# Sum of Areas

h	0.1	.2
2h	0.2	0.4
$2h^3/3$	0.0006666666666666	0.0053333333333333
$2h^5/5$	0.000004	0.000128
$2h^7/7$	0.0000000285714286	0.00000365714285714
$2h^9/9$	0.0000000002222222	0.000000113777777778
$2h^{11}/11$	0.00000000000018182	0.00000000372363636
$2h^{13}/13$	0.00000000000000154	0.00000000012603077
$2h^{15}/15$	0.00000000000000001	0.00000000000436907
Sum of Areas	0.200670695462151	0.405465108108002



# Difference of Areas

<b><math>h</math></b>	<b>0.1</b>	<b>.2</b>
<b><math>h^2</math></b>	<b>0.01</b>	<b>0.04</b>
<b><math>h^4/2</math></b>	<b>0.00005</b>	<b>0.0008</b>
<b><math>h^6/3</math></b>	<b>0.0000003333333333</b>	<b>0.00002133333333</b>
<b><math>h^8/4</math></b>	<b>0.0000000025</b>	<b>0.00000064</b>
<b><math>h^{10}/5</math></b>	<b>0.000000000002</b>	<b>0.00000002048</b>
<b><math>h^{12}/6</math></b>	<b>0.0000000000001667</b>	<b>0.00000000068267</b>
<b><math>h^{14}/7</math></b>	<b>0.0000000000000014</b>	<b>0.00000000002341</b>
<b>Diff'ce of Areas</b>	<b>0.010050335853501</b>	<b>0.040821994519406</b>

# The Area of the two separate regions(and related logarithms)

1/2 of the difference of these results and 1/2 of the sum of the results.

$$\ln(1.1) \approx 1/2 ( 0.2006706954621511 - 0.0100503358535014 )$$

$$\approx 0.0953101798043248$$

$$\ln(.9) \approx -(1/2)( 0.2006706954621511 + 0.0100503358535014 )$$

$$\approx -0.105360516578263 .$$

$$\ln(1.2) \approx 1/2 ( 0.405465108108002 - 0.040821994519406 )$$

$$\approx 0.18232155576939546 \text{ (from Newton)}$$

$$\ln(.8) \approx -(1/2)( 0.405465108108002 + 0.040821994519406 )$$

$$\approx -0.2231435513142097 \text{ (from Newton)} .$$

## Final calculations for $\ln(2)$

$$\begin{aligned}\ln(2) &= \ln\left(\frac{1.2}{.8} \frac{1.2}{.9}\right) = 2 \ln(1.2) - (\ln(.9) + \ln(.8)) \\ &\approx 2(0.18232155576939546) \\ &\quad + 0.105360516578263 \\ &\quad + 0.2231435513142097 \\ &= 0.6931471805599453 \text{ (from Newton)}\end{aligned}$$

# Comparison

$\ln(2)$  from Newton:

**0.6931471805599453**

$\ln(2)$  from calculator:

**0.69314718055994530941  
723212145818**

# From Newton

From Newton, *Of the Method of Fluxions and Infinite Series*, pp 132-133.

[Newton\\_on Pl.pdf](#)

ing these numbers for  $a$ ,  $b$ , and  $x$ , the first term of the series becomes 0.2, the second 0.0006666666666666, &c. the third 0.000004, and so on; as you see in this table.

$$\begin{array}{r}
 0.2000000000000000 \\
 6666666666666666 \\
 4000000000000000 \\
 285714286 \\
 2222222 \\
 18182 \\
 154 \\
 1
 \end{array}$$

$$0.2006706954621511 = \text{Area } bdDB.$$

If the parts of this Area  $Ad$  and  $AD$  be added separately, subtract the lesser  $DA$  from the greater  $dA$ , and there will remain  $\frac{bx^2}{a} + \frac{bx^4}{2a^3} + \frac{bx^6}{3a^5} + \frac{bx^8}{4a^7}$ , &c. where, if 1 be wrote for  $a$  and  $b$ , and  $\frac{1}{x}$  for  $x$ , the terms being reduced to decimals will stand thus.

$$\begin{array}{r}
 0.0100000000000000 \\
 5000000000000000 \\
 333333333 \\
 25000000 \\
 200000 \\
 1667 \\
 14
 \end{array}$$

$$0.0100503358535014 = Ad - AD.$$

Now if this difference of the Areas be added to, and subtracted from, their sum before found; half the aggregate 0.1053605156578263 will be the greater

greater Area  $Ad$ ; and half the remainder 0.0953101798043248 will be the lesser Area  $AD$ .

By the same tables these Areas  $AD$  and  $Ad$  will be obtained also, when  $AB$  and  $Ab$  are supposed 1.001, or  $CB=1.01$ , and  $Cb=0.99$ ; if the numbers are but duly transferred to lower places. As

$$\begin{array}{r}
 0.0200000000000000 \\
 6666666666 \\
 4000000 \\
 28 \\
 \hline
 \text{Sum } 0.020006667066695 = bD
 \end{array}
 \qquad
 \begin{array}{r}
 0.0001000000000000 \\
 50000000 \\
 3113 \\
 \hline
 0.0001000050003113 = Ad - AD
 \end{array}$$

Half the aggregate 0.0100503358535014 =  $Ad$  and Half the residue 0.0099503308531681 =  $AD$ .

And so putting  $AB$  and  $Ab = 1.001$ , or  $CB = 1.001$ , and  $Cb = 0.999$ , there will be obtained  $Ad = 0.00100050003335835$  and  $AD = 0.00099950013330835$ .

In the same manner (if  $CA$  and  $AF = 1$ ) putting  $AB$  and  $Ab = 0.2$ , or 0.02, or 0.002, these areas will arise.

$$\begin{array}{ll}
 Ad = 0.2231435513142097 & \text{and } AD = 0.1823215576939546 \\
 \text{or } Ad = 0.0201027073175194 & \text{and } AD = 0.0198026272961797 \\
 \text{or } Ad = 0.002002 & \text{and } AD = 0.001
 \end{array}$$

From these Areas thus found it will be easy to derive others by addition and subtraction alone, for as it is  $\frac{1.2}{0.8} \times \frac{1.2}{0.9} = 2$ ; the sum of the areas 0.6931471805599453 belonging to the ratios  $\frac{1.2}{0.8}$  and  $\frac{1.2}{0.9}$  (that is inscribing upon the parts of the absciss 1.2, 0.8. and 1.2, 0.9.) will be the area  $AF\delta\beta$ , when  $C\beta = 2$ ; as is known. Again, since  $\frac{1.2}{0.8} \times 2 = 3$ , the sum 1.0986122886681097

of

# Summary Analysis of Computation

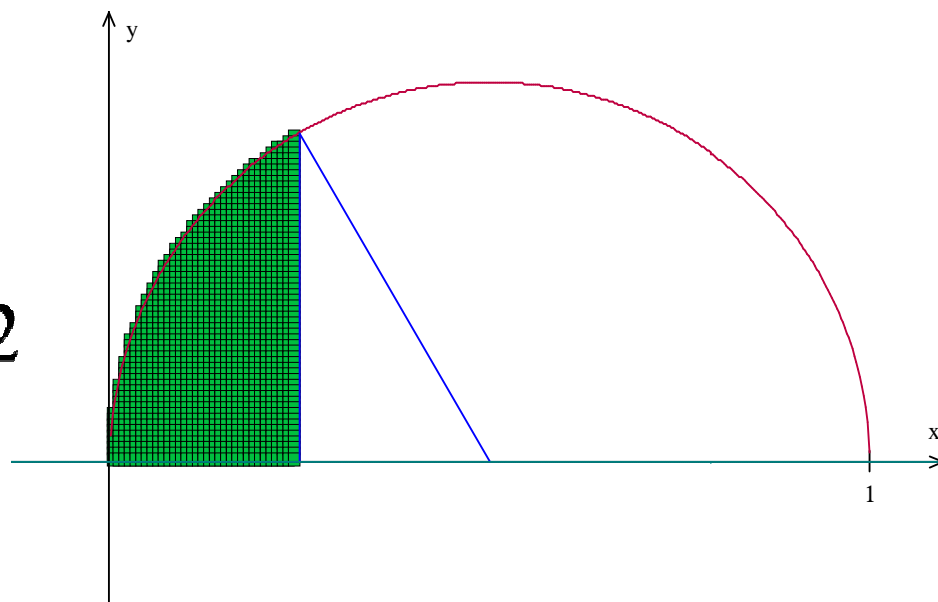
1. Use of geometric “series and polynomials” to estimate  $\frac{1}{1+x}$  and  $\frac{1}{1-x}$  when  $x \neq 0$ .
2. Integration of polynomials.
3. Geometry and algebra to decompose and recover estimates.
4. Algebra of logarithmic function.

$$\ln\left(\frac{A^2}{C \cdot D}\right) = 2\ln(A) - (\ln(C) + \ln(D)).$$

# Part II. Newton's computations of “ $\pi$ ”

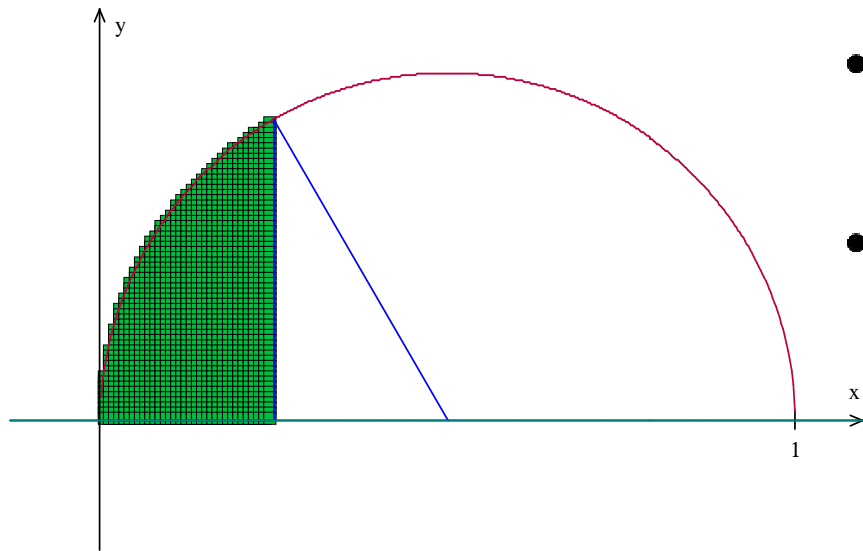
- Circumference of a circle is “ $2\pi r$ ”.
- Area of a circle is “ $\pi r^2$ ”.
- **Locate circle** of radius  $1/2$  with center at  $(1/2, 0)$ .
- **Equation for circle** is

$$y^2 = x(1 - x)$$
$$y = \sqrt{x} \sqrt{1 - x}$$





# Newton's Estimate of “ $\pi$ ”



- Use “series and polynomials” to estimate  $y = \sqrt{x} \sqrt{1-x}$ .
- Integrate “polynomials” to estimate area from 0 to  $1/4$ .
- Combine the area of the triangle from  $1/4$  to  $1/2$  to  $(1/4, \sqrt{3}/4)$  with the shaded area under the circle from 0 to  $1/4$  to cover the area of the central sector of  $1/6$ th of circle. This gives an estimate of “ $\pi/24$ ” to 16 places!

# Use of “Polynomials” and Integration

Polynomials used for  $\sqrt{1-x}$  : (Binomial Series)

$$\sqrt{1-x}: 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \dots$$

$$\sqrt{x} \sqrt{1-x}: x^{1/2} - \frac{x^{3/2}}{2} - \frac{x^{5/2}}{8} - \frac{x^{7/2}}{16} - \frac{5x^{9/2}}{128} - \frac{7x^{11/2}}{256} \dots$$

Now integrate to obtain:

$$\frac{2x^{3/2}}{3} - \frac{x^{5/2}}{5} - \frac{x^{7/2}}{28} - \frac{x^{9/2}}{72} - \frac{5x^{11/2}}{704} - \frac{7x^{13/2}}{1664} \dots$$

And for area of region under circle evaluate at  $\frac{1}{4}$  :

$$\frac{2(\frac{1}{2})^3}{3} - \frac{(\frac{1}{2})^5}{5} - \frac{(\frac{1}{2})^7}{28} - \frac{(\frac{1}{2})^9}{72} - \frac{5(\frac{1}{2})^{11}}{704} - \frac{7(\frac{1}{2})^{13}}{1664} = \frac{1}{12} - \frac{1}{160} \dots$$

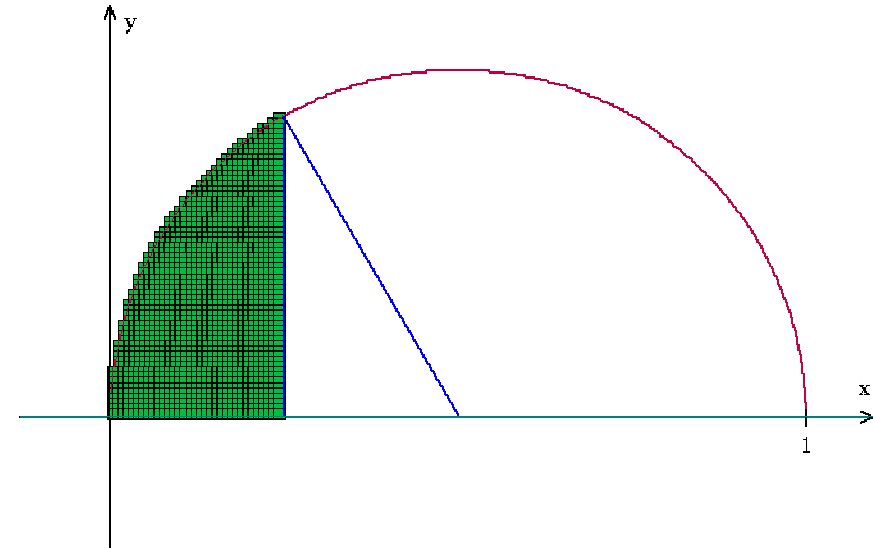
# Finale for Newton's estimate of “ $\pi$ ”

Area of Triangle:  $\frac{\sqrt{3}}{32}$ .

Area of central sector:  
 $\frac{1}{6}th$  of circle of radius  $\frac{1}{2}$

Area of circle (radius  $\frac{1}{2}$ )  $\approx$

$$6 \left( \frac{\sqrt{3}}{32} + \frac{1}{12} - \frac{1}{160} - \dots \right)$$



Circumference of circle (radius  $\frac{1}{2}$ ) = 4Area (=  $\pi$ )

$$\approx \textbf{(Newton)}(4)[6 \left( \frac{\sqrt{3}}{32} + \frac{1}{12} - \frac{1}{160} - \dots \right)] =$$

**3.1415926535897928**

# Comparison

$\pi$  from Newton:

3.14159265358979**28**

$\pi$  from calculator:

3.14159265358979**323846**  
**26433832795**

# From Newton

From Newton, *Of the Method of Fluxions and Infinite Series* , pp 130-131.

[Newton\\_on Pl.pdf](#)



$  \begin{array}{r}  352 \\  16 \\  1 \\  \hline  0.0896109885646618  \end{array}  $	$  \begin{array}{r}  0.0002825719389575 \\  + 0.0896109885646618 \\  \hline  0.0893284166257043  \end{array}  $
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Then from the sum of the affirmative, I take the sum of the negative terms, and there remains  $0.0893284166257043$  for the quantity of the Hyperbolick Area  $AdB$  which was to be found.

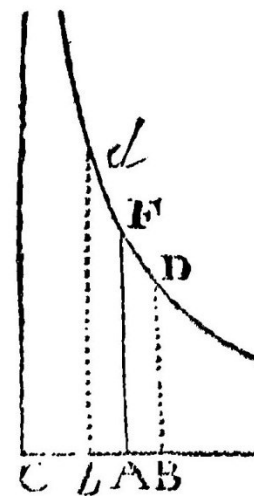
Let the Circle  $AdF$  [See the same Fig.] be proposed, which is expressed by the equation  $\sqrt{x-xx}=z$ , whose diameter is unity; and from what goes before its Area  $AdB$  will be  $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{7}x^{\frac{7}{2}} + \frac{1}{9}x^{\frac{9}{2}}$ , &c. in which series, since the terms do not differ from the terms of the series which above expressed the Hyperbolick Area, except in the signs  $+$  and  $-$ ; nothing else remains to be done, than to connect the same numeral terms with their signs; that is, by subtracting the connected sums of both the forementioned Tables,  $0.0898935605036193$ , from the first term doubled  $0.1666666666666666$ , &c. and the remainder  $0.0767731061630473$  will be the portion  $AdB$  of the Circular Area, supposing  $AB$  to be a fourth part

will give the Sextuple of which  $0.7853981634$  is the whole Area.

And hence (by the way) the length of the circumference will be  $3.141592653589793$  is found by dividing the Area by a fourth part of the diameter.

To this we shall add the calculation of the Area comprehended between the Hyperbola and its Asymptote  $CA$ , let  $C$

be the center of the Hyperbola, and putting  $CA=a$ ,  $AF=b$ , and  $AB=Ab=x$ ; it will be  $\frac{ab}{a+x} = BD$ , and  $\frac{ab}{a-x} = bd$ ; whence the Area  $AFDB = bx - \frac{bxx}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3}$ , &c. And the



$$\text{Area } AFdb = bx + \frac{bx^2}{2a} + \frac{bx^3}{3a^2} + \frac{bx^4}{4a^3}$$

$$\text{sum } bdDB = 2bx + \frac{2bx^3}{3a^2} + \frac{2bx^5}{5a^4} + \frac{2bx^7}{7a^6}$$

let us suppose  $CA=AF=1$ , and  $AB=Cb$  being  $=0.9$ , and  $CB=1.1$ . the



us reduced by degrees, I dispose into  
the affirmative terms in One, and the  
Another, and add them up as you

333333333333	— 0 . 0002790178571429
500000000000	34679066051
271267361111	834465027
5135169396	26285354
144628917	961296
4954581	38676
190948	1663
7963	75
352	4
16	
1	0 . 0002825719389575
	+ 0 . 0896109885646618
109885646618	0 . 0893284166257043

the sum of the affirmative, I take the  
negative terms, and there remains  
4166257043 for the quantity of the  
Area  $AdB$  which was to be found.

Circle  $AdF$  [ See the same Fig. ] be  
which is expressed by the equation  
whose diameter is unity; and from  
before its Area  $AdB$  will be  $\frac{2}{3}x^{\frac{1}{2}} - \frac{1}{5}x^{\frac{5}{2}}$   
 $\frac{1}{2}x^{\frac{7}{2}}$ , &c. in which series, since the terms  
fer from the terms of the series which  
pressed the Hyperbolick Area, except in  
and nothing else remains to be

part of the Diameter. And hence we may ob-  
serve, that though the Areas of the Circle and  
Hyperbola are not expressed in a Geometrical consi-  
deration, yet each of them is discovered by the  
same Arithmetical computation.

The portion of the Circle  $AdB$  being found, from  
thence the whole Area may be derived. For the ra-  
dius  $dC$  being drawn, multiply  $Bd$  or  $\frac{1}{4}\sqrt{3}$  into  $BC$   
or  $\frac{1}{4}$ , and one half of the product  $\frac{1}{2}\sqrt{3}$ , or  
 $0.0541265877365275$  will be the value of the  
Triangle  $CdB$ ; which added to the Area  $AdB$ ,  
will give the Sector  $ACd$ ,  $0.1308996938995747$ ;  
the Sextuple of which  $0.7853981633974482$   
is the whole Area.

And hence (by the way) the length of the Cir-  
cumference will be  $3.1415926535897928$ , which  
is found by dividing the Area by a fourth part of  
the diameter.

To this we shall add the calculation of the Area  
comprehended between the Hyperbola  $dFD$  and its  
Asymptote  $CA$ , let  $C$   
be the center of the Hy-  
perbola, and putting  
 $CA=a$ ,  $AF=b$ , and  
 $AB=Ab=x$ ; it will be  
 $\frac{ab}{a+x} = BD$ , and  $\frac{ab}{a-x}$   
 $=bd$ ; whence the Area  
 $AFDB = bx - \frac{bxx}{2a} + \frac{bx^3}{3a^2}$   
 $\frac{bx^4}{4a^3} - \frac{bx^6}{6a^4} + \frac{bx^8}{8a^5} - \frac{bx^{10}}{10a^6} + \frac{bx^{12}}{12a^7} - \frac{bx^{14}}{14a^8} + \frac{bx^{16}}{16a^9} - \frac{bx^{18}}{18a^{10}} + \frac{bx^{20}}{20a^{11}} - \frac{bx^{22}}{22a^{12}} + \frac{bx^{24}}{24a^{13}} - \frac{bx^{26}}{26a^{14}} + \frac{bx^{28}}{28a^{15}} - \frac{bx^{30}}{30a^{16}} + \frac{bx^{32}}{32a^{17}} - \frac{bx^{34}}{34a^{18}} + \frac{bx^{36}}{36a^{19}} - \frac{bx^{38}}{38a^{20}} + \frac{bx^{40}}{40a^{21}} - \frac{bx^{42}}{42a^{22}} + \frac{bx^{44}}{44a^{23}} - \frac{bx^{46}}{46a^{24}} + \frac{bx^{48}}{48a^{25}} - \frac{bx^{50}}{50a^{26}} + \frac{bx^{52}}{52a^{27}} - \frac{bx^{54}}{54a^{28}} + \frac{bx^{56}}{56a^{29}} - \frac{bx^{58}}{58a^{30}} + \frac{bx^{60}}{60a^{31}} - \frac{bx^{62}}{62a^{32}} + \frac{bx^{64}}{64a^{33}} - \frac{bx^{66}}{66a^{34}} + \frac{bx^{68}}{68a^{35}} - \frac{bx^{70}}{70a^{36}} + \frac{bx^{72}}{72a^{37}} - \frac{bx^{74}}{74a^{38}} + \frac{bx^{76}}{76a^{39}} - \frac{bx^{78}}{78a^{40}} + \frac{bx^{80}}{80a^{41}} - \frac{bx^{82}}{82a^{42}} + \frac{bx^{84}}{84a^{43}} - \frac{bx^{86}}{86a^{44}} + \frac{bx^{88}}{88a^{45}} - \frac{bx^{90}}{90a^{46}} + \frac{bx^{92}}{92a^{47}} - \frac{bx^{94}}{94a^{48}} + \frac{bx^{96}}{96a^{49}} - \frac{bx^{98}}{98a^{50}} + \frac{bx^{100}}{100a^{51}} - \frac{bx^{102}}{102a^{52}} + \frac{bx^{104}}{104a^{53}} - \frac{bx^{106}}{106a^{54}} + \frac{bx^{108}}{108a^{55}} - \frac{bx^{110}}{110a^{56}} + \frac{bx^{112}}{112a^{57}} - \frac{bx^{114}}{114a^{58}} + \frac{bx^{116}}{116a^{59}} - \frac{bx^{118}}{118a^{60}} + \frac{bx^{120}}{120a^{61}} - \frac{bx^{122}}{122a^{62}} + \frac{bx^{124}}{124a^{63}} - \frac{bx^{126}}{126a^{64}} + \frac{bx^{128}}{128a^{65}} - \frac{bx^{130}}{130a^{66}} + \frac{bx^{132}}{132a^{67}} - \frac{bx^{134}}{134a^{68}} + \frac{bx^{136}}{136a^{69}} - \frac{bx^{138}}{138a^{70}} + \frac{bx^{140}}{140a^{71}} - \frac{bx^{142}}{142a^{72}} + \frac{bx^{144}}{144a^{73}} - \frac{bx^{146}}{146a^{74}} + \frac{bx^{148}}{148a^{75}} - \frac{bx^{150}}{150a^{76}} + \frac{bx^{152}}{152a^{77}} - \frac{bx^{154}}{154a^{78}} + \frac{bx^{156}}{156a^{79}} - \frac{bx^{158}}{158a^{80}} + \frac{bx^{160}}{160a^{81}} - \frac{bx^{162}}{162a^{82}} + \frac{bx^{164}}{164a^{83}} - \frac{bx^{166}}{166a^{84}} + \frac{bx^{168}}{168a^{85}} - \frac{bx^{170}}{170a^{86}} + \frac{bx^{172}}{172a^{87}} - \frac{bx^{174}}{174a^{88}} + \frac{bx^{176}}{176a^{89}} - \frac{bx^{178}}{178a^{90}} + \frac{bx^{180}}{180a^{91}} - \frac{bx^{182}}{182a^{92}} + \frac{bx^{184}}{184a^{93}} - \frac{bx^{186}}{186a^{94}} + \frac{bx^{188}}{188a^{95}} - \frac{bx^{190}}{190a^{96}} + \frac{bx^{192}}{192a^{97}} - \frac{bx^{194}}{194a^{98}} + \frac{bx^{196}}{196a^{99}} - \frac{bx^{198}}{198a^{100}} + \frac{bx^{200}}{200a^{101}} - \frac{bx^{202}}{202a^{102}} + \frac{bx^{204}}{204a^{103}} - \frac{bx^{206}}{206a^{104}} + \frac{bx^{208}}{208a^{105}} - \frac{bx^{210}}{210a^{106}} + \frac{bx^{212}}{212a^{107}} - \frac{bx^{214}}{214a^{108}} + \frac{bx^{216}}{216a^{109}} - \frac{bx^{218}}{218a^{110}} + \frac{bx^{220}}{220a^{111}} - \frac{bx^{222}}{222a^{112}} + \frac{bx^{224}}{224a^{113}} - \frac{bx^{226}}{226a^{114}} + \frac{bx^{228}}{228a^{115}} - \frac{bx^{230}}{230a^{116}} + \frac{bx^{232}}{232a^{117}} - \frac{bx^{234}}{234a^{118}} + \frac{bx^{236}}{236a^{119}} - \frac{bx^{238}}{238a^{120}} + \frac{bx^{240}}{240a^{121}} - \frac{bx^{242}}{242a^{122}} + \frac{bx^{244}}{244a^{123}} - \frac{bx^{246}}{246a^{124}} + \frac{bx^{248}}{248a^{125}} - \frac{bx^{250}}{250a^{126}} + \frac{bx^{252}}{252a^{127}} - \frac{bx^{254}}{254a^{128}} + \frac{bx^{256}}{256a^{129}} - \frac{bx^{258}}{258a^{130}} + \frac{bx^{260}}{260a^{131}} - \frac{bx^{262}}{262a^{132}} + \frac{bx^{264}}{264a^{133}} - \frac{bx^{266}}{266a^{134}} + \frac{bx^{268}}{268a^{135}} - \frac{bx^{270}}{270a^{136}} + \frac{bx^{272}}{272a^{137}} - \frac{bx^{274}}{274a^{138}} + \frac{bx^{276}}{276a^{139}} - \frac{bx^{278}}{278a^{140}} + \frac{bx^{280}}{280a^{141}} - \frac{bx^{282}}{282a^{142}} + \frac{bx^{284}}{284a^{143}} - \frac{bx^{286}}{286a^{144}} + \frac{bx^{288}}{288a^{145}} - \frac{bx^{290}}{290a^{146}} + \frac{bx^{292}}{292a^{147}} - 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\frac{bx^{566}}{566a^{284}} + \frac{bx^{568}}{568a^{285}} - \frac{bx^{570}}{570a^{286}} + \frac{bx^{572}}{572a^{287}} - \frac{bx^{574}}{574a^{288}} + \frac{bx^{576}}{576a^{289}} - \frac{bx^{578}}{578a^{290}} + \frac{bx^{580}}{580a^{291}} - \frac{bx^{582}}{582a^{292}} + \frac{bx^{584}}{584a^{293}} - \frac{bx^{586}}{586a^{294}} + \frac{bx^{588}}{588a^{295}} - \frac{bx^{590}}{590a^{296}} + \frac{bx^{592}}{592a^{297}} - \frac{bx^{594}}{594a^{298}} + \frac{bx^{596}}{596a^{299}} - \frac{bx^{598}}{598a^{300}} + \frac{bx^{600}}{600a^{301}} - \frac{bx^{602}}{602a^{302}} + \frac{bx^{604}}{604a^{303}} - \frac{bx^{606}}{606a^{304}} + \frac{bx^{608}}{608a^{305}} - \frac{bx^{610}}{610a^{306}} + \frac{bx^{612}}{612a^{307}} - \frac{bx^{614}}{614a^{308}} + \frac{bx^{616}}{616a^{309}} - \frac{bx^{618}}{618a^{310}} + \frac{bx^{620}}{620a^{311}} - \frac{bx^{622}}{622a^{312}} + \frac{bx^{624}}{624a^{313}} - \frac{bx^{626}}{626a^{314}} + \frac{bx^{628}}{628a^{315}} - \frac{bx^{630}}{630a^{316}} + \frac{bx^{632}}{632a^{317}} - 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\frac{bx^{770}}{770a^{386}} + \frac{bx^{772}}{772a^{387}} - \frac{bx^{774}}{774a^{388}} + \frac{bx^{776}}{776a^{389}} - \frac{bx^{778}}{778a^{390}} + \frac{bx^{780}}{780a^{391}} - \frac{bx^{782}}{782a^{392}} + \frac{bx^{784}}{784a^{393}} - \frac{bx^{786}}{786a^{394}} + \frac{bx^{788}}{788a^{395}} - \frac{bx^{790}}{790a^{396}} + \frac{bx^{792}}{792a^{397}} - \frac{bx^{794}}{794a^{398}} + \frac{bx^{796}}{796a^{399}} - \frac{bx^{798}}{798a^{400}} + \frac{bx^{800}}{800a^{401}} - \frac{bx^{802}}{802a^{402}} + \frac{bx^{804}}{804a^{403}} - \frac{bx^{806}}{806a^{404}} + \frac{bx^{808}}{808a^{405}} - \frac{bx^{810}}{810a^{406}} + \frac{bx^{812}}{812a^{407}} - \frac{bx^{814}}{814a^{408}} + \frac{bx^{816}}{816a^{409}} - \frac{bx^{818}}{818a^{410}} + \frac{bx^{820}}{820a^{411}} - \frac{bx^{822}}{822a^{412}} + \frac{bx^{824}}{824a^{413}} - \frac{bx^{826}}{826a^{414}} + \frac{bx^{828}}{828a^{415}} - \frac{bx^{830}}{830a^{416}} + \frac{bx^{832}}{832a^{417}} - \frac{bx^{834}}{834a^{418}} + \frac{bx^{836}}{836a^{419}} - \frac{bx^{838}}{838a^{420}} + \frac{bx^{840}}{840a^{421}} - \frac{bx^{842}}{842a^{422}} + \frac{bx^{844}}{844a^{423}} - \frac{bx^{846}}{846a^{424}} + \frac{bx^{848}}{848a^{425}} - \frac{bx^{850}}{850a^{426}} + \frac{bx^{852}}{852a^{427}} - \frac{bx^{854}}{854a^{428}} + \frac{bx^{856}}{856a^{429}} - \frac{bx^{858}}{858a^{430}} + \frac{bx^{860}}{860a^{431}} - \frac{bx^{862}}{862a^{432}} + \frac{bx^{864}}{864a^{433}} - \frac{bx^{866}}{866a^{434}} + \frac{bx^{868}}{868a^{435}} - \frac{bx^{870}}{870a^{436}} + \frac{bx^{872}}{872a^{437}} - \frac{bx^{874}}{874a^{438}} + \frac{bx^{876}}{876a^{439}} - \frac{bx^{878}}{878a^{440}} + \frac{bx^{880}}{880a^{441}} - \frac{bx^{882}}{882a^{442}} + \frac{bx^{884}}{884a^{443}} - \frac{bx^{886}}{886a^{444}} + \frac{bx^{888}}{888a^{445}} - \frac{bx^{890}}{890a^{446}} + \frac{bx^{892}}{892a^{447}} - \frac{bx^{894}}{894a^{448}} + \frac{bx^{896}}{896a^{449}} - \frac{bx^{898}}{898a^{450}} + \frac{bx^{900}}{900a^{451}} - \frac{bx^{902}}{902a^{452}} + \frac{bx^{904}}{904a^{453}} - 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# Analysis of Computation

1. Use of binomial series polynomials to estimate  $\sqrt{1-x}$  Then multiplied by  $\sqrt{x}$ .
2. Integration of “series ... polynomials”.
3. Geometry and algebra to decompose and recover estimates.
4. Algebra of geometric areas:

Area of sector = area of triangle + area under circle.

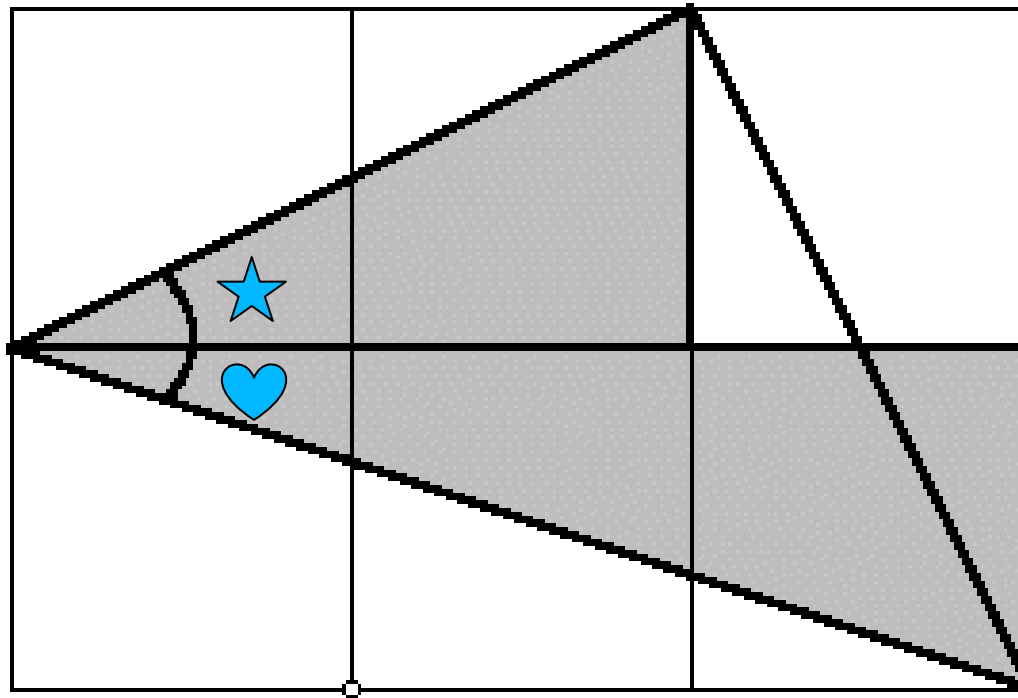
## Part III.

Newton's logarithmic scheme for computations applied to estimating “ $\pi$ ” .  
Basic Scheme:

- Use polynomials as geometric series for  $\frac{1}{1+x^2}$  .
- Integration of  $\frac{1}{1+x^2}$  polynomials gives polynomial for  $\arctan(x)$ .
- Use values “close to 0.”
  - $\arctan(1/2)$ ;  $\arctan(1/3)$
- Use addition reductions.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}.$$

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$$



# Note on Other Estimates of $\pi$

John Machin (1706-**Jones**) : 100 places

William Shanks (1873): 707 places- 527 correct!

$$4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) = \frac{\pi}{4}.$$

Euler: adopted  $\pi$  as a symbol.

Developed other equations for estimation such as

$$5\arctan\left(\frac{1}{7}\right) + 2\arctan\left(\frac{3}{79}\right) = \frac{\pi}{4}$$


Current best (?): Alexander J. Yee & Shigeru  
Kondo (August 2, 2010) 5 trillion places.

# Exercise for Calculus

Apply the analysis from Newton's estimate for  $\ln(2)$  to create an estimate for " $\pi$ " from the arctan identity...

# Estimate of “ $\pi$ ” with Excel

k	n=2k-1	$(-1)^{(k+1)}x^n/n$ x=1/2	$(-1)^{(k+1)}x^n/n$ x=1/3
1	1	0.500000000000000000000000	0.3333333333333333000000
2	3	-0.0416666666666666700000	-0.012345679012345700000
3	5	0.006250000000000000000000	0.000823045267489712000
4	7	-0.001116071428571430000	-0.000065321052975374000
5	9	0.0002170138888888889000	0.000005645029269476760
6	11	-0.000044389204545454500	-0.000000513184479043342
7	13	0.000009390024038461540	0.000000048248113414331
8	15	-0.000002034505208333330	-0.000000004646114625084
9	17	0.000000448787913602941	0.000000000455501433832
10	19	-0.000000100386770148026	-0.000000000045283768276
11	21	0.000000022706531343006	0.000000000004552336493
12	23	-0.000000005183012589164	-0.000000000000461831238
13	25	0.000000001192092895508	0.000000000000047209415
14	27	-0.000000000275947429516	-0.000000000000004856936
15	29	0.000000000064229143077	0.000000000000000502442
	<b>Estimate</b>	<b>0.463647609012972000000</b>	<b>0.321750554396642000000</b>
	<b>arctangent</b>	<b>0.463647609000806000000</b>	<b>0.321750554396642000000</b>
		<b>pi estimate = 4(atan(1/2)+atan(1/3))</b>	<b>3.14159265363846</b>
		<b>pi</b>	<b>3.14159265358979</b>

The End  
  
Questions?

[flashman@humboldt.edu](mailto:flashman@humboldt.edu)  
[users.humboldt.edu/flashman](https://users.humboldt.edu/flashman)