Square Roots: Adding Philosophical Contexts and Issues to Enhance Understanding.

Preliminary report.
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Abstract

- The nature of numbers can be confusing to students in a variety of learning contexts. One frequently encountered area of confusion surrounds numbers described as square roots, such as the square root of 2 and the square root of -1. The author will examine how illuminating some philosophical approaches to the nature of numbers (ontology) and knowledge about numbers and their properties (epistomology) [can] **may** help students avoid some possible confusion.
- Time permitting the author may suggest possible empirical studies for (college level) students to provide evidence for the utility of introducing more philosophical approaches to pedagogy.

Philosophy of Mathematics

Currently most philosophy of mathematics has two major concerns:

- Ontology
- Epistemology

Philosophy of Mathematics in a Nutshell I

- **Ontology** for Mathematics: "Being"
- Ontology studies the nature of the objects of mathematics.

  What is a **number**?

  What is a **point**? **line**?

  What is a **set**?

  In what sense do these objects exist?

Philosophy of Mathematics in a Nutshell II

- Epistemology for Mathematics: “Knowing”
- Epistemology studies the acquisition of knowledge of the truth of a mathematical statement.
  - Does knowledge come from experience and evidence?
  - Does knowledge come from argument and proof?
  - Is knowledge relative or absolute?
  - How does ontology influence knowledge?

Encounters with Squares and Roots

From a PRE-ALGEBRA Text – found in the chapter on decimals.

Once you've mastered the process of squaring a whole number, then you are ready for the inverse of the squaring process, taking the square root of a whole number. Above, we saw that $9^2 = 81$. We called the number 81 the square of the number 9.

Conversely, **we call the number 9 a square root of the number 81.** Above, we saw that $(-4)^2 = 16$. We called the number 16 the square of the number −4. Conversely, **we call the number −4 a square root of the number 16.**

Square Root. If $a^2 = b$, then a is called a square root of the number b.

... Because $(-3)^2 = 9$ and $3^2 = 9$, both −3 and 3 are square roots of 9. **Special notation, called radical notation, is used to request these square roots.**

..... Not all numbers are perfect squares. For example, in the case of $\sqrt{24}$, there is no whole number whose square is equal to 24. **<u>However, this does not prevent $\sqrt{24}$ from being a perfectly good number.</u>**

- **<u>Important Observation. A calculator can only produce a finite number of decimal places. If the decimal representation of your number does not terminate within this limited number of places, then the number in your calculator window is only an approximation.</u>**

Encounters with Squares and Roots

- From a Beginning Algebra Text:
- Since π and √2 cannot be written as fractions with an integer numerator and a nonzero integer denominator, they are not rational numbers. They are called irrational numbers. … If we combine the rational and the irrational numbers, we have the set of **real numbers.**
- **In a Box: The Pythagorean Theorem applied to an isosceles right triangle with leg 1 and hypotenuse of length c, gives "an example of an aspect of the natural world that corresponds to an irrational number, namely c = √2 ."**
- The number b is a square root of a if $b^2 = a$.
- All positive numbers have square roots. …
  Square roots of certain numbers such as 7 are hard to compute by hand. However we can find an approximation … with a calculator

Encounters with Squares and Roots

From Precalculus Text:

"Real numbers are used throughout mathematics and you should be acquainted with symbols that represent them, such as …

...Real numbers that are not rational are **irrational numbers**. … There is *no rational* number b^2 such that $b^2 = 2$... However there is an *irrational* number, denoted by $\sqrt{2}$… such that $(\sqrt{2})^2 = 2$."

"Real numbers may be represented by points on a line... " …

Definition... If a is a real number and $a > 0$, then $\sqrt{a}$ is the *positive* real number b such that $b^2 = 2$

… The .. laws … are true...*provided the individual roots* ***exist*** – that is , provided the roots are real numbers.

Complex numbers are needed to find solutions of equations that cannot be solved using only … real numbers.... The letter i does not represent a real number. It is a new mathematical entity that will enable us to obtain the complex numbers.

We must consider … expressions of the form $a + bi$ ...

... *we may treat all symbols as having properties of real numbers*,...

Encounters with Squares and Roots

- From Calculus Text Appendix:
- “Some real numbers, such as $\sqrt{2}$, can't be expressed as a ratio of integers, and are therefore called **irrational numbers**.
- *…* Every [real] number has a decimal representation.
- ... Real numbers can be represented by points on a line... … Often we identify the point with its coordinate and think of a number as being a point on the real line.
- A **complex number** can be represented by an expression of the form $a + bi$ ... i is a symbol with the property that $i^2 = -1$. The complex number $a + bi$ can also be represented by the ordered pair (a , b)and plotted as a point on the plane ... are needed to find solutions of equations that cannot be solved using only … real numbers.... The letter i does not represent a real number. It is a new mathematical entity that will enable us to obtain **C**....
  We must consider … expressions of the
  ... *we may treat all symbols as having properties of real numbers*,...

Encounters with Squares and Roots
The square root of 2

- **Theorem:** The square root of two is not a rational number (is an irrational number).
- Common **Proof Outline**: [An indirect proof by contradiction.]
  Assume it is , = m/n; so 2 $*n^2$ = m^2.... use arithmetic of natural numbers to arrive at a contradiction. [Many ways to do this.]

Encounters with Squares and Roots
The square root of -1

- **Theorem**: The square root of negative one is not a real number (is a pure imaginary number).
- Common **Proof Outline**: [An indirect proof by contradiction.]
  Assume it is a real number, call it "i ".
  So $i^2 = -1$.... use arithmetic of real numbers to arrive at a contradiction. [Many ways to do this.]

Questions arising from these “proofs”

- Why do we care about square roots?
- How do we know these are numbers? That these numbers exist?
  What makes these “objects” numbers?
- How do we gain knowledge about these “objects”?

Why do we care about square roots?

- Geometry: We are interested in shapes, their components and measurement: Squares and the sides of squares- “square roots”. We measure these objects with numbers (magnitudes).
- Algebra - arithmetic: We are interested in numbers and their operations: Solving number equations of the form $x^2 = a$, i. e., finding the “square root of a”. Numbers are represented with symbols, notational systems - numerals, decimals, etc and visualized through geometry.
- The geometry and algebra problems are connected to “real world” problems we wish to solve.

How do we know these are numbers? That these numbers exist?

- **Geometry:** Once a unit of length is determined, a number corresponds to a length. The number is independent of the unit. Its representation is contingent on the unit. [Euclid/Descartes]

  Once a unit segment is established, the measurement of a line segment is a number.
  Example: The square root of two.

- **Algebra/Arithmetic:** A natural number is determined from a unit, or is an equivalence class of sets, or... All numbers arise from operations and properties (possibly infinite) describing and relating numbers.
  Example: The square root of -1.

- Other examples: pi, e, ln(2)

What makes these “objects” numbers? Ontological Commitment

- The object may have a primary ontological commitment (justification) from geometry or algebra. This is chosen or evolved.
- Secondary ontological commitments may be given as alternatives to assist understandings and beliefs.

Examples of Ontological Commitments (OC)

- Natural numbers (Integers):
  - Geometric: A unit line segment, multiple unit line segment (oriented) extensions.
  - Algebraic: ordinal arithmetic, Peano arithmetic, Set theory equivalence classes, (signed) decimal Arabic numerals
- Rational numbers
  - Geometric: Oriented line segments with multiples being integers.
  - Algebraic: Results of “division”, solving equations; ratio(nal) arithmetic; terminal and repeating decimals.

Examples of Ontological Commitments (OC)

- Square root of two:
  - The primary ontological commitment is geometric.
  - Secondary commitment is algebraic: Solving an equation; an infinite decimal; a limit of a sequence of rational numbers; an equivalence class of cauchy sequences....
- Square root of -1:
  - The primary ontological commitment is algebraic.
  - Secondary commitment is geometric: Using three non co-linear points to determine the three numbers: 0, 1, and i.

How do we gain knowledge about these “objects”?

- Epistemology and Ontology.
- We gain knowledge from
  - Intuition- a priori
  - Experience (unification)
  - Induction (generalization)
  - Reason (Logical deduction and Argument)
  - Construction
  - Authority

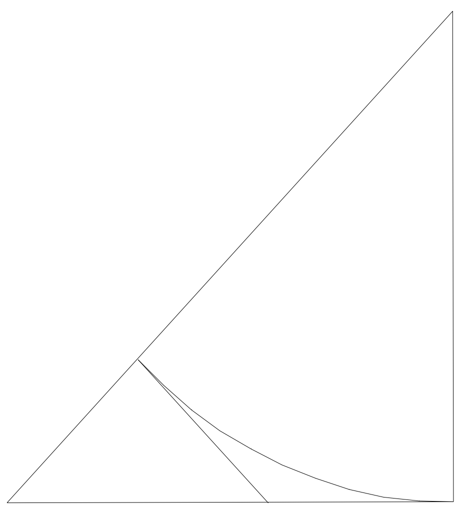
Connecting Epistemology & Ontology

- Failure to recognize how epistemology and ontology are related can undercut the credibility of a learning experience
  - It removes context from statements and objects.
  - It encourages formalism and/or authority as the basis for statements.
- Connecting epistemology to ontology reinforces a meaningful learning experience.
  - It provides context for statements and objects.
  - It encourages belief in the connectedness of knowledge through a web of experience and reason.

Example: Proof that $\sqrt{2}$ is not a rational number based on geometric ontology.

Figure based on *Irrationality of The Square Root of Two -- A Geometric Proof,* Tom M. Apostol, The American Mathematical Monthly, Vol. 107, No. 9 (Nov., 2000), pp. 841-8

- Suppose $\sqrt{2}$ is rational, then there is a **smallest** isosceles right triangle with positive integer sides.
  [Rescaling with unit fixed.]
- This leads to a contradiction.



Example: Proof that $\sqrt{(-1)}$ is not a real number based on algebraic ontology.

Use the algebraic properties of the real numbers as an ordered field to show that the square of any real number is a real number that is either 0 or positive.

There is a field extension of the field of real numbers where the equation $x^2 = -1$ has a solution. In this field extension, the solution is not a real number.

Conclusion?

- Recognition of the ontological commitment of a mathematical object can
  - help **connect epistemological issues with ontology** and thereby
  - enhance learning by **making arguments and experience relevant to understanding the nature of the object.**

Conclusion applied?

- The common approach to the irrationality of the square root of two could be improved by using a geometric argument that connects to the primary ontology for the number.
- The common approach to the nature of the square root of -1 can be enhanced by recognizing that algebraic ontology is used for other numbers before the introduction of this example where it will serve as the primary ontology.

Other root issues?

- Geometry: Construction of roots with straight edge and compass.
  Theorem: If a is constructable, then square root of a is constructable.
  The cube root of 2 is not constructable.
- Algebra: Given *a* in a field, to construct a field extension that contains a solution to $x^n = a$.
  Theorem: If a is in a field, then there is a field extension that contains a solution to $x^n = a$.
  The cube root of 2 is not constructable.
  - The square root of pi ? Squaring the circle.

References

- *Tom M. Apostol, Irrationality of The Square Root of Two -- A Geometric Proof,* The American Mathematical Monthly, Vol. 107, No. 9 (Nov., 2000), pp. 841- 842.
- http://www.cut-the-knot.org/proofs/sq_root.shtml

  Many proofs that square root of 2 is irrational.

Questions?
Comments?
Discussion?

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This presentation will be linked
on the web
by Jan.15 through

http://users.humboldt.edu/flashman

The End.

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