Do You Want Your Students To Write Proofs? Suggestions to Improve Writing Proofs

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Availability

These slides and links will be available at my web home page:

- http://users.humboldt.edu/flashman/

- Or search using “Flashman” and “math”.

Acknowledgement

This work is based in part on many years of experiences teaching at HSU: **Math 240 (3 units) Introduction to Mathematical Thought and Math 381 (1 unit) Tutorial in Writing Proofs**

Primary Resource Texts:

Daniel Solow's **How To Read and Do Proofs (Solow 1) and**

The Keys to Advanced Mathematics... (Solow 2)

How To Solve It by G. Polya,

Starter: Euclid Book I Proposition 1

Problem:

Construct an equilateral triangle on a given finite straight line.

Understanding the Problem:

Course Preparations

1. Know Your Students.

1. Attitudes?
2. Beliefs?
3. Commitment?
4. Experience?
5. Habits?
6. Knowledge?

Most likely-your students are not your clones!

1.1 Time for The Serenity Prayer

Repeat with me:

- … grant me serenity to accept the things I cannot change.
- Courage to change the things I can.
- And the wisdom to know the difference.

2. First Step(s) for Students

- Step 1. I am a mathematician. [Attitude. Belief.]
- Step 2. There are methods for determining mathematical truth. [Belief. Experience.]
- Step 3. I can master these method through practice, openness, and willingness. (POW) [Attitude. Belief. Commitment. Habit.]
- Step 4. I will acknowledge the challenges for me to become a better mathematician through a personal inventory of strengths and weaknesses. [Attitude. Belief. Commitment. Experience. Habit. Knowledge.]
- Step 5. I will keep working (by myself and with others - including my instructor) to become a better mathematician. [Attitude. Commitment.]

3. Consider The Big Picture: Read Polya

"How To Solve It" by G. Polya.

4 Phases of Problem Solving

1. Understand the problem.
2. See connections to devise a plan.
3. Carry out the plan.
4. Look back. Reflect on the process and results.

Planning

4.1 The Big Picture: Organizing A Course on Proofs

Consider Options

do's and don'ts

4.1 The Big Picture: Organizing A Course on Proofs (I)

Avoid lengthy discussion of traps

- Logic and truth tables.
  [Remember: This is not a course in Logic.]
- Venn diagrams and proving set equalities.
  [Remember: This is not a course in Set Theory.]
- Mathematical Induction.
  [Remember: Most problems are not solved easily.
  The induction step is usually not a proof by induction.]

4.2 The Big Picture: Organizing A Course on Proofs (II)

Cover mathematics that illustrates Mathematical Thinking:

- Arithmetic . Primes, division, and factors.
  [Remember: This is not a course in Number Theory.]
- Rational, real and complex numbers. Operations, order, open sets.
  [Remember: This is not a course in calculus, analysis, or topology]
- Finite and infinite sets.
  [Remember: This is not a course in set theory.]
- Functions: Discrete and continuous; specific and abstract.
  [Remember: This is not a discrete math nor a precalculus course.]
- Counting: finite and infinite.
  [Remember: This is not a course in combinatorics.]

4.3 The Big Picture: Organizing A Course on Proofs (III)

Focus on <u>organization</u>-

Recognize generic "proof schemes" and "key questions": (Solow and others)

- Brief look at "logic"
- Look for connections: "key questions" , definitions, and theorems!
- Conditional statements: "Assume… show…"
- Universal statements: "Choose … show…"
- Existential statements: "Construct … show…"
- Indirect Arguments: Contrapositive & Contradiction
- Special techniques:
  - Induction / Well Ordering Principle (Use of natural numbers and order.)
  - Uniqueness
  - Alternatives

Context

It is important to understand the role that **context** plays in understanding the interpretation and meaning of mathematical statements and proofs.

Read Bertrand Russell's landmark philosophical analysis of the use of language in "On Denoting" (Mind, 1905).

5. Use a Refined Concept of "Generalization" in Relation to Context

- Solow, Daniel. (1995). *The Keys to Advanced Mathematics: Recurrent Themes in Abstract Reasoning.*
- **See** Flashman, "Projection: Examples to Illuminate Unification, Generalization, and Abstraction." (on-line, 2013)
- Unification: A single context.
- Generalization: Distinct but connected, related contexts.
- Abstraction: Broad context definition with a structural characterization that allows discourse with limited specificity.

Executing your plan:
Some practical suggestions

6. Composition Activities

Engage students in

- Transforming Arguments
- Proof Analysis
- Examining Evidence

Composition: Learning to write ....to write proofs

- **Articulation and transformation** of understanding: a skill used in initial creation and development of a composition.[Describe a scene, relate the objects, explain the connections.]

- **Analysis and deconstruction** of a composition: a skill used in reading compositions. [Recognize the outline, parse a sentence, define a word.]

6.1 Engage Students in Transforming Arguments

- **Task: Articulation and transformation** of understanding: a skill used in initial creation and development of a composition.
  - The exercise prompts the student to transform a figure that encompasses **nonverbal thoughts and arguments** (“proofs without words”) into a readable **verbal presentation of the related argument** or "proof."

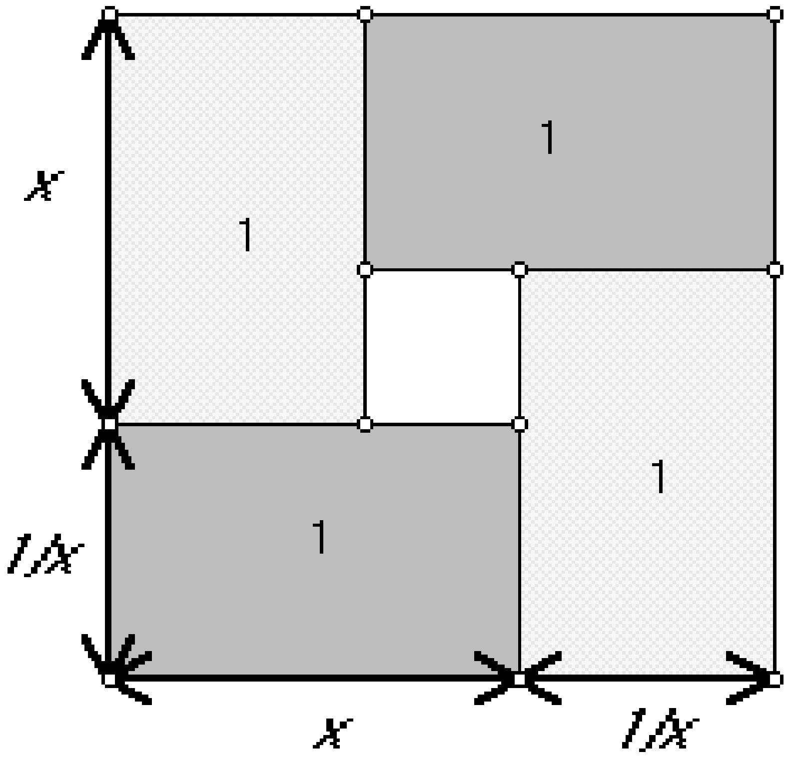
Proof without Words

With a Partner discuss and use **Figure 1** to "prove" :

$$\boldsymbol{x + 1/x \geq 2}.$$

One partner(A) should write the "proof" while the other partner(B) checks to see the writing fully reports their understanding.

Figure 1



6.2 Engage students in Proof Analysis

- **Task: Analyze and deconstruct** a composition:

  proof-reading.
- Make a **systematic structural and content** analysis of a given proof to understand the presentation.
  - Overview: What is the basic organization? Who is the intended audience?
  - Correctness: Is the argument sound? Definitions? Logic?
  - Coherence and Readability: Did the presentation of the argument make sense? Were there reader road signs? Omissions?
  - Alternatives: What might have improved the argument?

A Mathematician's Habit of Thought: Examining Evidence to Understand Statements and Proofs

- Mathematicians have a habit of thought in **giving interpretation-meaning to statements in contexts**.
- This habit allows the mathematician to **understand initially a statement or proof** and focus attention on **the material meaning by testing evidence**.
- Without this habit, a student's initial understanding of a statement **may be deficient in interpreting the content meaning in an appropriate context**.
- Without initial understanding, the student can proceed too quickly to making a plan with resulting confusion from being inadequately prepared.

6.3 Engage Students in Examining Evidence

- **Task: Examine evidence to form an u**nderstanding: a skill used in initial creation and development of a composition.
  - The exercise prompts the student to connect words with interpretations providing evidence for or against the truth of statements .

Some examples of work intended to develop a habit.

Examining evidence **to attach meaning to the words.**

Suppose $X = \{1, 3, 5, a, c, e\}, Y = \{1, 2, 3, a, b, c\}, Z = \{2, 4, 6, b, d, f\}$ and $W = \{4, 5, 6, d, e, f\}$

- $Is\ 1\ a\ member\ of\ X?$ ___ $Y?$ ___ $Z?$ ___ $W?$ ___
- $Is\ 2\ an\ element\ of\ X?$ _ $Y?$ $Z?$ $W?$
- $Is\ a \in X?$ _ $Y?$ $Z?$ $W?$
- $Is\ b \in X?$ _ $Y?$ $Z?$ $W?$
- List all sets that have 3 as an element.
- List all elements that are **members of both X and Y.**
- List all elements that are **members of either X or Y.**
- List all elements of X **that are not elements of Y.**
- List all elements of the set $Z \cap W$.
- List all elements of the set $Z \cup W$.
- List all elements of the set $Z - W$.

Some examples of work intended to develop a habit.

Examining evidence **to attach meaning to statements.**

Suppose **E** = { n: n is an integer and there is an integer k where $n = 2k$};

O = { n: n is an integer and there is an integer k where $n = 2k + 1$}

T = { n: n is an integer and there is an integer k where $n = 3k$}

For each of the following conditional statements (if possible) **give separate examples of a number x (i) where the hypothesis is true; (ii) where the conclusion is true; (iii) where the hypothesis is false; (iv) where the conclusion is false.**

Do you believe the statement is true or false?

a) If x is a member of E then x^2 is a member of E.

b) If x is a member of O then x^2 is a member of O.

c) If x is a member of T then x^2 is a member of T.

d) If x is a member of T then $x + 1$ is a member of E.

e) x is a member of T only if $x^2 + x$ is a member of E.

Reflection
What more?

Should Philosophy Play a Larger Role in Learning about Proofs?

Euclid Book I Proposition 1

To construct an equilateral triangle on a given finite straight line.

Proof: Given finite straight line AB.

With center A construct circle O with radius AB.

With center B construct circle O' with radius AB.

Construct Segment AC from A to C, the point of intersection of O and O'.

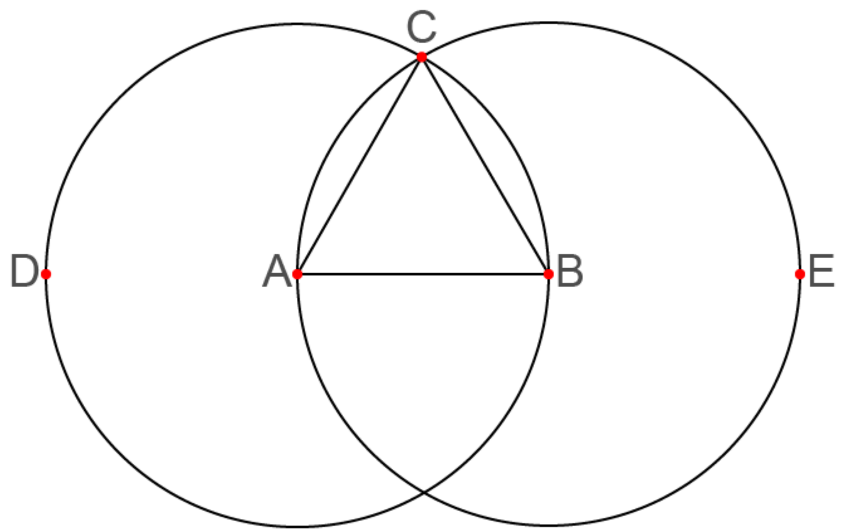
Construct Segment BC from B to C, the point of intersection of O and O'.

AC = AB.

BC = AB.

The triangle ABC is the desired equilateral triangle.

QEF.



7. Recognize Some Philosophy in Proofs

- Understand the context: As in philosophy- questions are often more important than answers.
  - What do the words mean?. What do the sentences mean?
  - Platonism, formalism, structuralism, empiricism
  - Example: The positive square root of 3 is not a rational number.
- Befriend the stranger-Existence meets philosophy: **Semantics and Ontological Commitment** (Quine)
  - The empty set: $\{(p,q)\colon\ p^2 = 3\,q^2, p, q\ \in N^+\}$
  - The least upper bound of $\{x \in Q :\ x^2 < 3\}$
  - Infinite sets. $\{x \in Q :\ x^2 < 3\}$
- Recognize the value of a monster: **Proofs and Refutations** (Lakatos)
  - The square root of 3.
  - The absolute value function.
  - The Cantor set
  - The Russell paradox

References

- Solow, Daniel. (1995). *The Keys to Advanced Mathematics: Recurrent Themes in Abstract Reasoning*. Cleveland Heights, Ohio: Books Unlimited.
- Solow, Daniel. (2014). *How to Read and Do Proofs: An Introduction to Mathematical Thought Processes*, *6th Edition*. New York, Wiley. [Look inside on Amazon.com]

Previous presentations on how to write proofs

- Two Different Approaches to Getting Students Involved in Writing Proofs. (JMM, *January*, 2011)
- Understanding the Problem: Unification, Generalization or Abstraction? (JMM, *January*, 2013)
- The Benefits of A Habit: Examining Evidence to Understand Statements and Proofs. (*JMM*, *January*, 2013)
- Logic is Not Epistemology: Should Philosophy Play a Larger Role in Learning about Proofs? (MathFest, *August*, 2013)