Functions, Duality, and Mapping Diagrams

State of Jefferson Math Congress

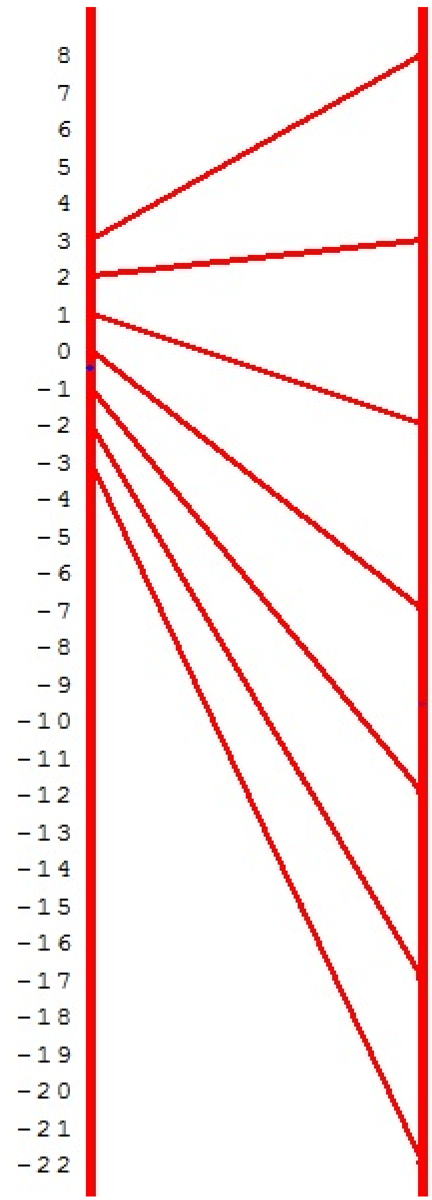
Oct. 5, 2013

Martin Flashman

Professor of Mathematics
Humboldt State University
flashman@humboldt.edu
http://users.humboldt.edu/flashma

What is a Mapping Diagram?
What happens before the graph.

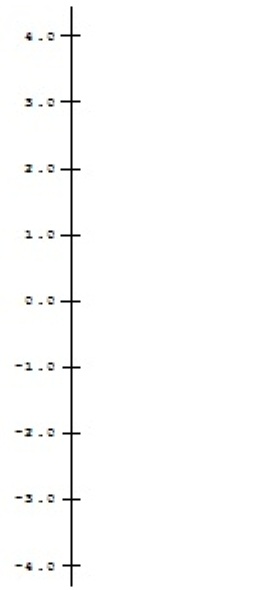
X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22



A.1 Suppose f is a function determined by the following table :

t	-4	-3	-2	-1	0	1	2	3	4
f (t)	-3	-2	0	3	4	3	2	-1	0

Complete the following mapping diagram for f with the indicated numbers. [Use the same scale for the second axis.]

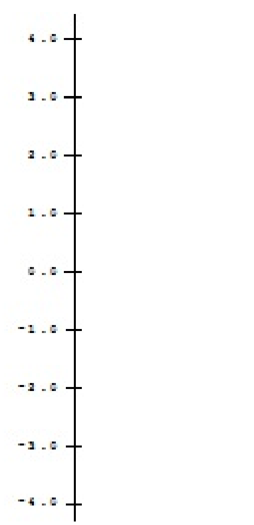


B. Suppose that $f(x) = 2x + 1$ for all $x \in \mathbb{R}$.

B.1. Complete the following table :

x	f(x)
2	
1	
0	
-1	
2	

B.2. Complete the following mapping diagram for f with the indicated numbers. [Use the same scale for the second axis.]



Worksheet C

$f(x) = mx + b$ Sketch mapping diagram.

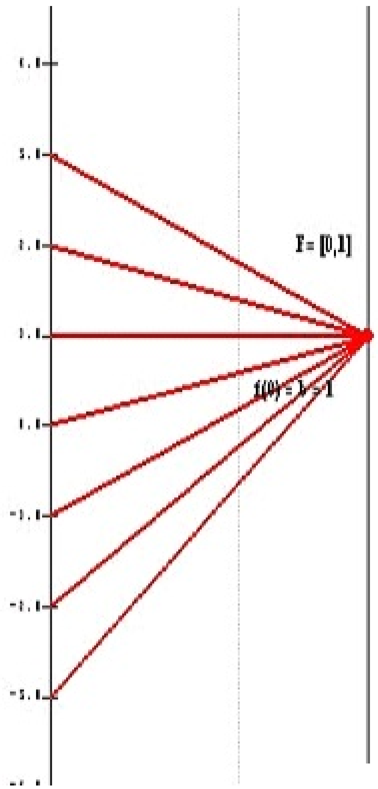
1: $m = -2$; $b = 1$: $f(x) = -2x + 1$

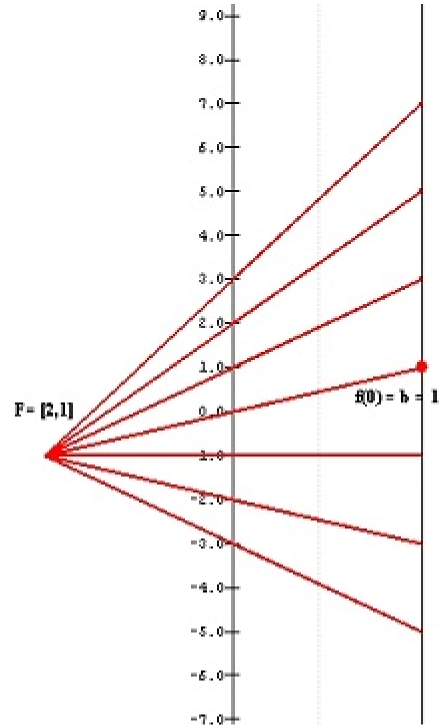
2: $m = 2$; $b = 1$: $f(x) = 2x + 1$

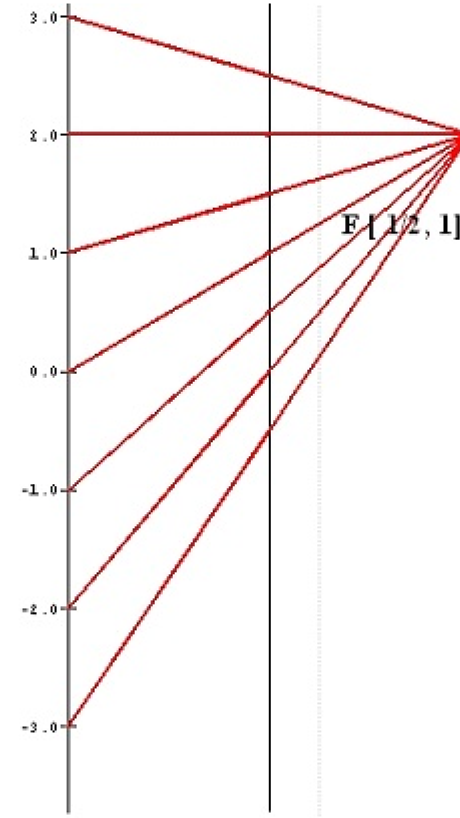
3: $m = \frac{1}{2}$; $b = 1$: $f(x) = \frac{1}{2}x + 1$

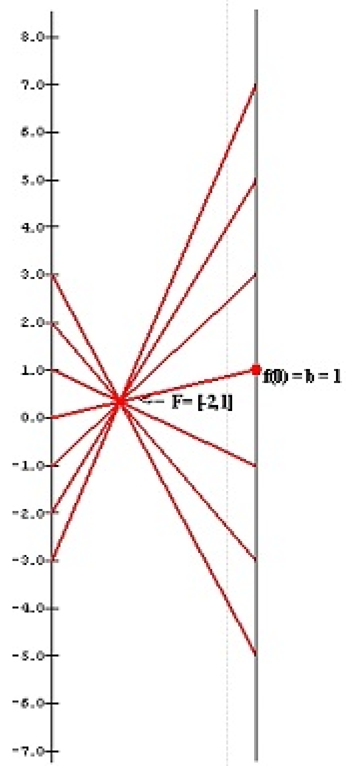
4: $m = 0$; $b = 1$: $f(x) = 0x + 1$

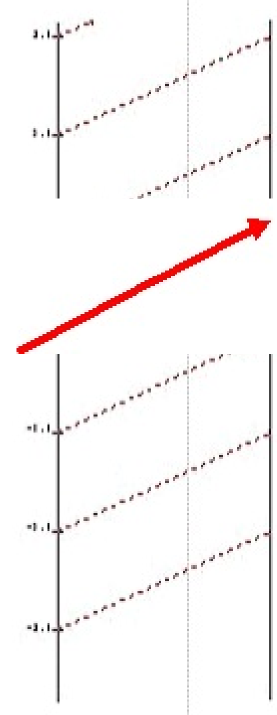
5: $m = 1$; $b = 1$: $f(x) = x + 1$


F= [0,1]
f(0) = b = 1


F= [2,1]
f(0) = b = 1


F [1/2 , 1]


F= [-2,1]
f(0) = b = 1



Duality in Geometry

Duality: A pairing of words, concepts and figures.

Words

Point	→	**Line**
Line	→	**Point**
Intersection	→	**Join**
Join	→	**Intersection**

Concepts and Figures

Triangle	→	Trilateral
Graph of Function	→	Mapping Diagram
A range of points	→	A pencil of lines

Duality Exercise

I say “**point**”; you say “**line**”.

I say “**line**”; you say “**point**”.

I say “**intersection**”; you say “**join**”.

I say “**join**”; you say “**intersection**”.

Principle of Duality in Projective Planar Geometry (PPG, also RP^2)

Suppose **S** is a statement in PPG and **S'** is the corresponding dual statement in PPG that results by applying the appropriate changes to the words and concepts of S.

Principle: S is a theorem of PPG if and only if S' is a theorem of PPG.

Application of Duality in PPG

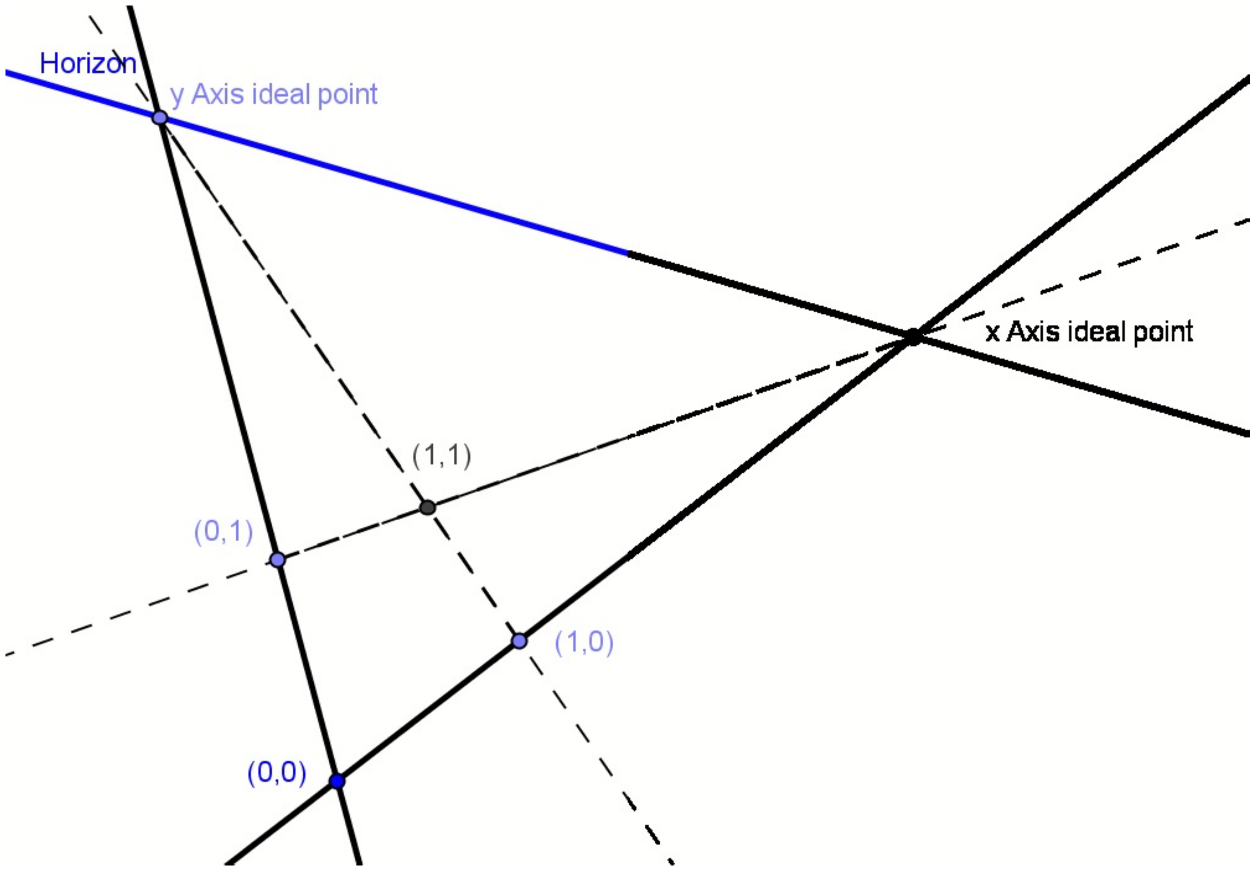
S: Two points, P and Q, determine a unique line, the join of the points P and Q.

S′: Two lines, p and q, determine a unique point, the intersection of the lines p and q.

- When lines are parallel in Euclidean Geometry they meet in PPG at a unique **point at infinity**.
- Two distinct points at infinity determine a unique line, the line at infinity (the "horizon" line).

So **parallel lines meet on the horizon line.**

A look at RP^2 visualizing PPG.



Look at Worksheet E.1.

Graph $f(x) = x$ and $g(x) = x+1$ in RP^2.

Duality and Linear Functions

Suppose f: R → R is defined by

$$\mathbf{f(x) = mx+b.}$$

The graph of f,

$$\ell_f = \{ (x,y):\ y = f(x)\ ,\ x \in R\}$$

is a unique line in the euclidean plane, which we can consider a line in PPG.

A point P in Euclidean *geometry* lies on ℓ_f if and only if **for some a** $\in$ **R, P is the point of intersection of two lines**

$$\{(a,y);\ y \in R\} \text{ and } \{(x,f(a):\ x \in R\}.$$

Consider ℓ_f as a line in PPG.

Then by the principle of duality, there is a unique point, $\mathbf{L_f}$, in PPG with dual properties:

$\mathbf{L_f}$ has a distinguished pencil of lines passing through it determined by the function f.

A line p passes through $\mathbf{L_f}$, if and only if for some $a \in R$, p is the join of two points:

A, a point determined on a line (the X axis) corresponding to the number a, and

B=f(A), a point determined on a line (the Y axis) corresponding to the number f(a).

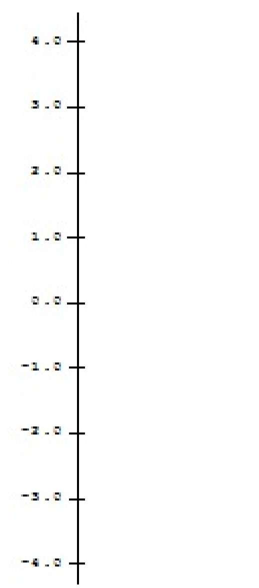
We will call the point $\mathbf{L_f}$ **the "focus point" for the linear function f.**

Applications of the Focus Point.

1. Two pairs of data determine a linear function, f. (1,3), (2,0). Find f(0)

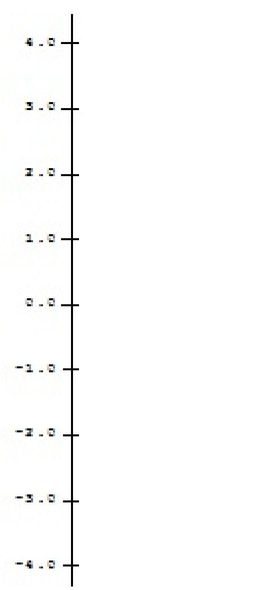
Solution: Draw the arrows on the Mapping Diagram and extend them to meet at the Focus Point.

To find f(0) draw the line from the focus point on the Domain Axis on the MD. Where the line meets the Target axis is f(0).



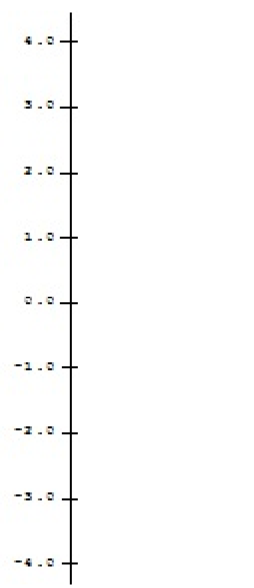
2. Using the same function as in the previous problem find a where f(a)= -6.

Solution: Draw the line from the focus point of f to -6 on the target axis. Where this line meets the Domain axis, is the desired value for a.



3. Suppose $g(x)=m'x+b'$ is a linear function with $m' \neq m$. Find a where $f(a) = g(a)$.

Solution outline: Find the focus points for f and g. Draw the line determined by these two focus points. Where it meets the Domain axis is the desired number a.



Visualizing functions f: $RP^1 \to RP^1$

Many $f: R \to R$ functions extend to give functions from RP^1 to RP^1.

Examples:

1. $f(x) = 4$: $f(\infty) = 4$.

2. $f(x) = 3x + 4$: $f(\infty) = \infty$.

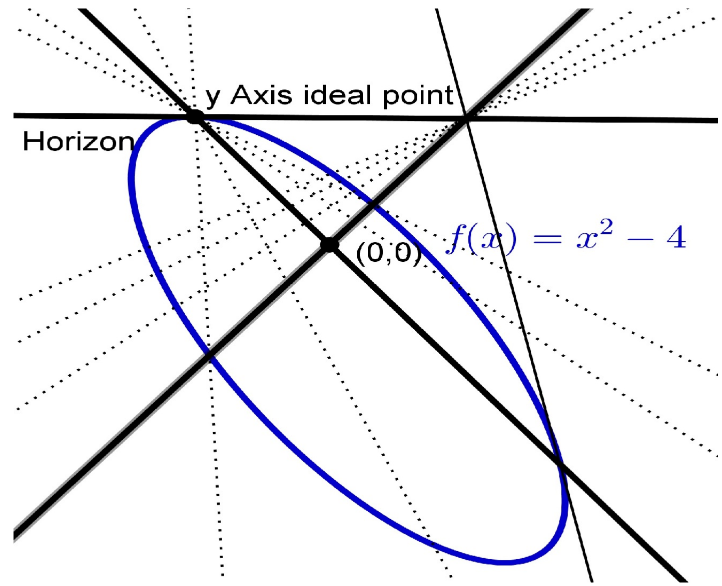
3. $f(x) = x^2 - 4$: $f(\infty) = \infty$.

4. $f(x) = 1/(x-1)$: $f(1)=\infty$. $f(\infty) = 0$.

We can visualize these in RP^2 by treating the infinite values appropriately.

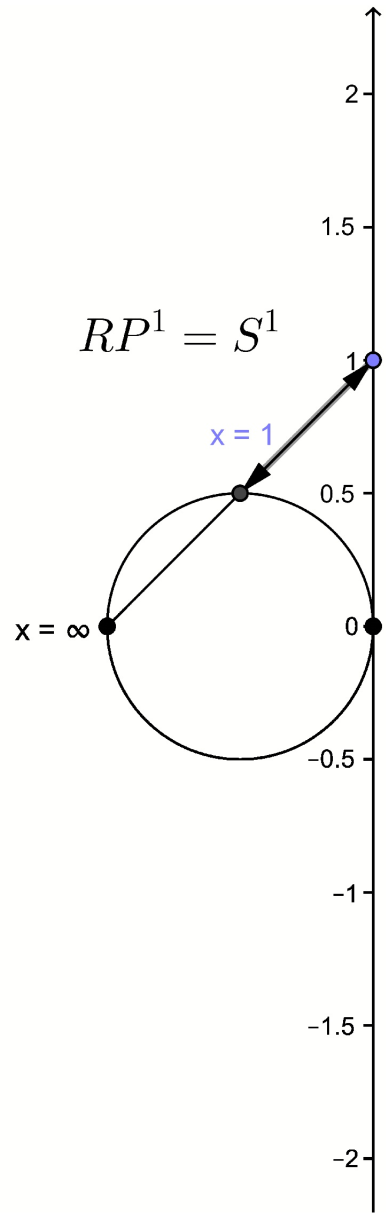
1. $f(x) = 4$, $f(\infty) = 4$ (?).
2. $f(x) = 3x + 4$, $f(\infty) = \infty$.
3. $f(x) = x^2 - 4$. $f(\infty) = \infty$.

[Created with Geogebra]



4. $f(x) = 1/(x-1)$. $f(1)=\infty$. $f(\infty) = 0$.

But in fact RP^1 can be visualized as a circle.
See Worksheet D.

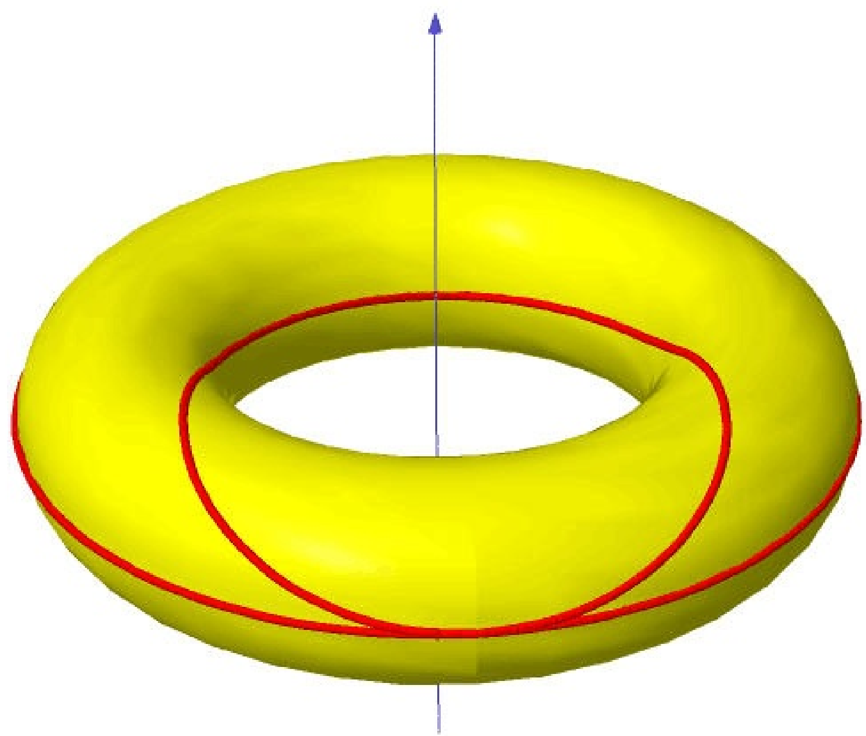


Using this we have two more ways to visualize a function $f: RP^1 \to RP^1$:

- The graph of $f: RP^1 \to RP^1$ is a curve on $RP^1 \times RP^1$ - the torus!

Example: $f(x) = x^2$. $f(\infty) = \infty$.

[Created with SAGE]



- The mapping diagram of f, a "surface" with boundaries a circle and a subset of a circle.

Example: $f(x) = 1/(x^2-1)$. $f(\pm 1)=\infty$. $f(\infty) = 0$.

[Created with SAGE]