Logic is Not Epistemology: Should Philosophy Play a Larger Role in Learning about Proofs? WORK IN PROGRESS

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Abstract

Many **transition to proof (TP) courses** start with a review or introduction to what is often described as "logic".

The author suggests that **students might be better served with an alternative approach that connects notions of proof with philosophical discussions related to ontology and epistemology**.

Examples will be offered to illustrate some possible changes in focus.

Preparation Questions

- How many of you have taught a transition to proof course?
- How many of you have taught "logic" with truth tables for propositions and venn diagrams for quantification?
- How many of you discuss some aspect of the philosophy of mathematics in your teaching?

Polya's 4 Phases of Problem Solving

1. **Understand the problem.**
2. **See connections** to devise a plan.
3. Carry out the plan.
4. Look back. **Reflect on the process and results.**

Mathematics and Logic

Mathematical proof of a conditional statement is not identical to a demonstration involving only truth tables and the syntax of quantification.

Material implication is used in mathematics because **in mathematics the concern is focused primarily on contexts where the connection of statements in conditional statements has significant meaning**.

The Examples (as time permits): Consider 6 Statements and Proofs

- Euclid Book I Proposition 1
  - *To construct an equilateral triangle on a given finite straight line.*
- Euclid Book IX Proposition 20
  - *Prime numbers are more than any assigned multitude of prime numbers.*
- Pythagoras (?):
  - *The square root of 2 is not a rational number.*
- Cantor:
  - *The rational (or algebraic) numbers are equi-numerous with the natural numbers.*
- Cantor:
  - *The real numbers (or points on a line segment) are infinite but not equi-numerous with the natural numbers.*
- Russell:
  - *R* = { *S: S is not an element of S* } *is not a set.*

Euclid Book I Proposition 1

To construct an equilateral triangle on a given finite straight line.

Proof: Given finite straight line AB.
With center A construct circle O with radius AB.
With center B construct circle O' with radius AB.
Construct Segment AC from A to C, the point of intersection of O and O'.
Construct Segment BC from B to C, the point of intersection of O and O'.
AC = AB.
BC = AB.
The triangle ABC is the desired equilateral triangle.
QEF.

Euclid Book I Proposition 1

To construct an equilateral triangle on a given finite straight line.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interests:
- Construction is existence. QEF vs QED
- Definitions based on primitives.
- Euclid Axioms built to model "reality".
- Hilbert approach to (formal) axioms for geometry.

Missing assumption:

The existence of point of intersection of circles.
- The power of counterexamples: Proofs and refutations (Lakatos)
- Alternative (models for) geometry :
  - Rational geometry.
  - Geometry without compass but with Playfair parallel postulate.

Euclid Book IX Proposition 20

Prime numbers are more than any assigned multitude of prime numbers.

Proof:

Suppose the primes comprise p1, p2, …, pn.

Let q = p1*p2* … *pn + 1.

Then q is not a prime.

But any number is either a prime or has a prime factor.

So one of the primes, p1, p2, … , pn, is a factor of q.

But that same prime is a factor of p1*p2* … *pn so it must be a factor of 1. This is absurd, so

The primes are more than any assigned multitude of prime numbers.

QED.

Euclid Book IX Proposition 20

Prime numbers are more than any assigned multitude of prime numbers.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interests:

- Existence without construction. QED (not QEF)
- Definitions and prior results in an information web. (Structures)
- Euclid definitions built to generalize multiple measurement contexts : length, area, volume.
- Peano axioms abstract structure and "implication" relationship. Russell- Whitehead build from abstract logic.
  Other foundations for numbers based on set measurement and equivalence relations.

Importance of consistency:

- Mathematics abhors contradiction within its structures.
- Indirect proof and construction depend on consistency.

Pythagoras

The square root of 2 is not a rational number.

Proof:

Suppose r is a rational number and $r^2=2$.

r = a/b where a, b are positive natural numbers.

Then rb=a and $r^2 b^2 = a^2$.

Or $2 b^2 = a^2$.

Counting the number of 2 factors of the right hand side: Even .

Counting the number of 2 factors of the left hand side: Odd .

This contradicts the FT of Arithmetic. So the square root of 2 is not a rational number.

Q.E.D.

Pythagoras

The square root of 2 is not a rational number.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interests:

- Definitions of rationality emphasized.
  - Definitions and prior results in an information web. (Structures)
    - Equality of rational numbers and uniqueness of representation.
    - Prime number factors.
  - Equality and operations are context dependent.
- Algebraic Axioms or set theoretic construction of rational numbers. What is a fraction?
  - Existence usually ignored (presumed). QED
  - Existence usually presumed for numbers as measures from geometric model.
  - Numbers solve equations.
  - Existence of Algebraic Numbers:
    - What about the square root of -1.
    - What about π?

Cantor and Countability

***The rational (or algebraic) numbers are equi-numerous with the natural* numbers.(Cantor/Godel)**

Proof: The set of rational numbers contains a subset equi-numerous with the natural numbers, so the set of rational numbers is infinite.

Suppose a natural number N of the form $N=2^k*3^m*5^n$ where k=0 or 1, m, n are natural number and n≠0.

The map *f* is defined for N to a rational number

***f*(N) = m/n if k=0 and *f*(N) =-m/n if k = 1.**

By definition of the rational numbers, this map is onto.

So we have a map from an infinite subset of the natural numbers onto the rational numbers.

Thus the rational numbers are equi-numerous with the natural numbers.

Cantor and Countability

The rational (or algebraic) numbers are equi-numerous with the natural numbers.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interests:

- Existence by construction. QEF [not QED]
- Definitions and prior results in an information web. (Structures)
- Like proof by first diagonal argument of Cantor: requires removal of redundancy due to multiple representations of rational numbers.
- **Does not construct bijection.**
- Definition of rational numbers as equivalence classes.

Cantor and Uncountability

The real numbers (points on a line segment) are infinite but not equi-numerous with the natural numbers.

Proof:

Suppose there is a bijection, F, from the natural numbers to the real numbers.

Let b be the real number b= 0.b1b2b3b4…. Where

b1 = 7 if the first decimal digit of F(1) ≠7 and b1=5 if the first decimal digit of F(1) = 7.

b2 = 7 if the second decimal digit of F(2) ≠7 and b2=5 if the second decimal digit of F(2) = 7.

bk = 7 if the kth decimal digit of F(2) ≠7 and bk=5 if the kth decimal digit of F(2) = 7.

If b = F(n) then bn is not the nth decimal digit of b. This is absurd by the definition of b, so F is not onto the real numbers. Thus the real numbers are not equi-numerous with the natural numbers.

Cantor and Uncountability

The real numbers (points on a line segment) are infinite but not equi-numerous with the natural numbers.

Philosophical interests:

- Existence by hypothetical construction. QEF and QED
- Definitions and prior results in an information web. (Structures)
- This proof combines with others to prove the existence of transcendental numbers indirectly.
- What is a real number?
  - An equivalence class of sequences of rational numbers. (Cauchy sequences)
  - A set of rational numbers with special properties (A Dedekind cut)
  - Any element of a model for a complete ordered field of characteristic 0.
- How is the Continuum Hypothesis an axiom of Set Theory?

Russell

R = { *S: S is not an element of S* } *is not a set.*

Proof:

Suppose R is a set.

Then the statement: "R is an element of R" is a proposition.

If R is an element of R, then by definition of R, R is not an element of R .

This is a contradiction, so R is not an element of R.

Now by definition of R, R is an element of R.

In summary, if R is a set there is a mathematical proposition that is a contradiction.

This is absurd. So R is not a set.

Russell

R = { *S: S is not an element of S* } *is not a set.*

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Set theory needs some restrictions to be free from contradictions.

Can mathematics be analyzed by mathematics alone?

Is there a need for some philosophy to understand
- mathematical objects ?
- mathematical structures ?
- mathematical knowledge ?

Logic is Not Epistemology

Should Philosophy Play a Larger Role in Learning about Proofs?

Answer:

?

The End

- Thanks-
- Questions

These slides will be posted by Sept. 1 at http:// users.humboldt.edu/flashman