Mapping Diagrams: Four Sessions, One Theme. Using Mapping Diagrams in Algebra, Calculus, and Complex Analysis

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Introduction

Mapping diagrams were the central theme for four sessions presented at BCME9. A mapping diagram is a visual tool for understanding functions and equations that provides clarity for several procedures and concepts that are not easily treated with a Cartesian graph. The argument (input) of a function - an element of the domain (source) set – is visualised as a point on a number line (axis). Separately the corresponding value of the function (output) - an element of the codomain (target) set is visualised on a second parallel axis. An arrow from the input to the output visualises the relation between the two numbers, see figure 1.

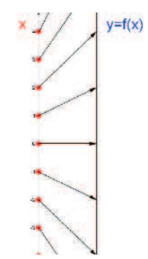


Figure 1.
Mapping Diagram

A mapping diagram can display many arrows comparable to a table, or the point in the domain can be moved with graphic technology giving a dynamic visualisation of the function. A similar mapping diagram can be created for functions of complex numbers by replacing the two axes with two planes visualising the complex domain and co-domain, see figure 2.

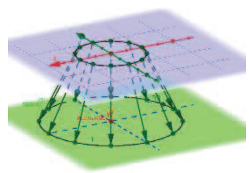


Figure 2.
Mapping Diagram for Complex Function

The mapping diagrams in the figures and used in the four sessions were created using GeoGebra. The content of each session will be discussed separately in what follows. Links to the GeoGebra Books for all the sessions, references, and many other related materials can be found at the website: users.humboldt.edu/flashman/Presentations/BCME/BCME.LINKS.html.

Visualising Quadratic, Cubic, and Quartic Equation Solutions: An Introduction to Complex Numbers, Functions, and Mapping Diagrams

Solving equations is one of the early and dominant tasks presented to students as they begin the study of algebra. Starting with linear equations, students learn to solve the general quadratic equation and some more accessible cubic and quartic equations with little or no visualisation provided. A visual approach is less available when complex numbers are introduced to solve even the simple quadratic equation $x^2 + 1 = 0$. In this session mapping diagrams of functions are used in two ways: first to visualise the steps in the algebraic solution of linear and quadratic equations by recognising the functions in these equations as compositions and then to visualise the nature of complex solutions to quadratic, cubic, and quartic equations.

For example solving the linear equation $2x + 1 = 5$ is connected to the composite mapping diagram for the functions $m(x) = 2x$ and $s(x) = x + 1$. The

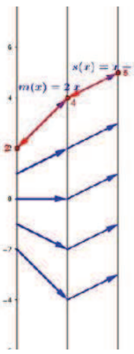


Figure 3. Mapping Diagram Solving Linear Equation

number 5 is represented by a point on the final target axis, and the algebraic steps are visualised by reversing the direction of the arrows (inverting the functions) of the composition to arrive at the point/number solution, $x = 2$ on the first source axis, see figure 3.

A quadratic equation represented in the "vertex" form, for example, $2(x-1)^2+1=9$ also is connected to a mapping diagram for a composite of four core functions and the algebraic steps for solving the equations are visualised similarly when real roots exist (see figure 4) and later even for complex roots using complex mapping diagrams.

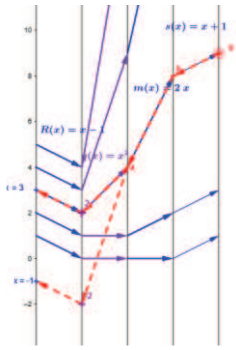


Figure 4. Mapping Diagram Solving Quadratic Equation

Cubic, and quartic equations are not solved algebraically in the school curriculum but can also exhibit complex number roots. For these equations, mapping diagrams for complex functions can visualise dynamically important connections between the roots and the coefficients of the polynomial functions. For example the following mapping diagrams (figures 5 and 6) visualise the cubic equation $x^3 - 1 = 0$ has one real root and two complex conjugate roots, and the equation $x^4 + 1 = 0$ has two pairs of complex conjugate roots.

Figure 5. Mapping Diagram Solving a Cubic Equation

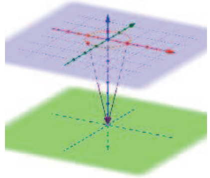
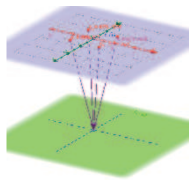


Figure 6. Mapping Diagram Solving a Quartic Equation



How Many Ways Can You Solve a Quadratic Equation Visually? From the Greeks to 21st Century Technology

Quadratic equations have a long history, going back at least to the Greeks and Euclid's Elements. The visual solution of these equations starts with finding the defining segment (root) for a square with the same area as a given rectangle and the Pythagorean Theorem for finding the root for the sum of two squares. Quadratic equations are visualised as an unknown square with an attached rectangle being equal in area to a given square. The ancient solution of this problem used the completion of the square in a literal sense, see figure 7.

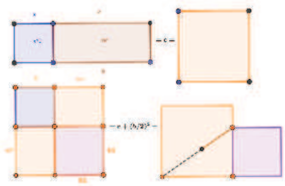


Figure 7. Completing the Square: Quadratic

Later Descartes revolutionised the problem by treating the problem as one about lengths of segments where the product of segments is visualised by another segment, see figure 8. The solution of a quadratic equation is one of the first uses Descartes makes of this visualisation, solving the problem without any explicit mention of geometric squares - though still completing the square with a construction similar to that of the ancients.

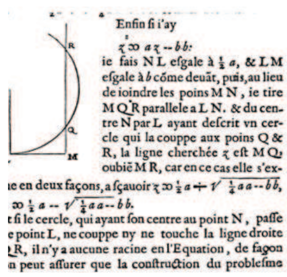


Figure 8. Descartes Solution of a Quadratic Equation

The use of the graph of a quadratic function to determine the axis of symmetry by shifting the graph visualises the process of completing the square, but does not show the actual algebraic steps for solving the equation after that step. The algebraic steps are visualised by the mapping diagrams as treated above in the section on solving equations.

Making Sense of Integration Visually: Mapping Diagrams for Calculus

The tradition of the 20th century was to link the definite integral with finding the area of a region in the plane bounded by the graph of a function on a compact interval. An alternative treatment connects the definite integral to solving a differential equation and the net change in any solution's value for a compact interval. This second approach is closely connected to the dynamics of change and motion that motivated Napier in his definition of the logarithm and Newton in his understanding of fluxions. A visualisation based on this dynamics approach to the definite integral is nicely done by using mapping diagrams.

Initially the mapping diagram visualises linear functions as a magnification with a focus point, see figure 9. The magnification factor for the linear function that defines the derivative of a function is then the model for estimating the function for a small interval about a number of interest, $x=a$. This is the differential estimator, $dy = df(a,dx) = f'(a)dx$. This is also visualised with a mapping diagram showing that $f(a+dx) \approx f(a) + dy = f(a) + f'(a)dx$, see figure 10.

Figure 9. Mapping Diagram for $f(x) = 2x + 1$

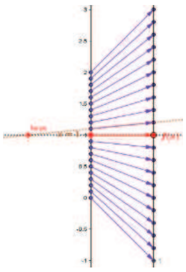
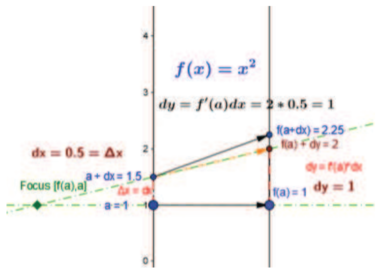


Figure 10. Mapping Diagram for differential: $f(x)=x^2$, $x=1$, $dx=0.5$



To estimate the net change for solving a differential equation $f'(x) = P(x)$ over the interval $[a,b]$ we use Euler's method with initial condition $f(a) = 0$. The estimated net change will be a sum of differentials $\sum f'(x_k)\Delta x = \sum P(x_k)\Delta x$. The accumulation of these differentials can be visualised with mapping diagrams, so the sum estimates the net change in the value of a solution to the differential equation, $f(b) - f(a)$, see figure 11.This difference is independent of the choice of the initial value.

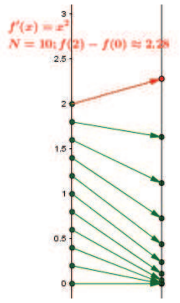


Figure 11. Mapping Diagram Estimate of net change: Accumulated Differentials

The limit of these sums defines the definite integral, and so the evaluation of the definite integral $\int_a^b P(x)dx$ is the net change in any solution to the differential equation, $f'(x) = P(x)$ *i.e.*, we have a definition of the definite integral and immediately one form of the Fundamental Theorem of Calculus for continuous functions: $\int_a^b P(x)dx = f(b) - f(a)$ with f being any continuous function where $f'(x) = P(x)$ for all x in $[a,b]$.

Making Sense of Complex Analysis with Mapping Diagrams: A New Visualisation Tool Enhanced by Technology

It is often said that the task of complex analysis is to generalise the calculus of real variables to a calculus for complex variables. The first difficulty encountered in complex analysis after visualising complex numbers by identifying them with points in a coordinate plane, with $a + bi$ corresponding to the point (a,b), and making sense of arithmetic for complex numbers is to visualise functions, linear and nonlinear elementary functions. Because the real calculus emphasises the graph of the function as the main visualisation, at this stage the difficulty of a graph for complex functions is primarily that of visualising a four dimensional Euclidean space where the graph of a complex function would appear as quadruple of real numbers- (x, y, u, v): $where\ f(x + iy) = u + iv$. By using mapping diagrams to visualise and understand real functions, the visualisation of complex functions can be handled comparably without needing "four dimensions".

Linear functions are the key to both differentiation and integration of complex functions as they were for real functions. Mapping diagrams for complex multiplication arise from an amplification from a focus point – the vertex of a cone and a twist-rotation resulting from the angle determined by the complex number multiplier, see figure 12.

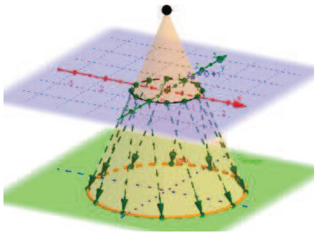


Figure 12. Mapping Diagram for $f(z) = (2+i)z$ applied to the circle $|z|=1$.

The magnification factor for the linear function that defines a complex function's derivative is the model for estimating that function for a small disc about a number of interest, $z = a + bi$, that is the differential estimator, $dw = df(a + bi, dz) = f'(a + bi)dz$. This is visualised with a complex function mapping diagram showing $f(a + bi + dz) \approx f(a + bi) + dw = f(a + bi) + f'(a + bi)dz$ in figure 13, $f(z) = z^2, a + bi = i, and\ dz = 1 + 0.5i$.

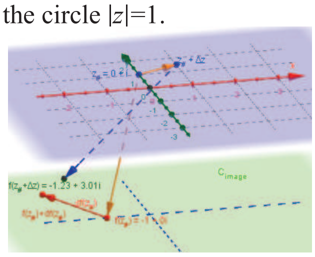


Figure 13.Mapping Diagram for Differential: $f(z)=z^2$, $z=i$, $dz=1+0.5i$.

The line integral for complex analysis is determined along a curve γ in the complex plane given parametrically by estimating the net change for solving a differential equation $f'(z) = P(z)$ over the defining interval $[a, b]$ for the curve's parameters. As with the real integral we use Euler's method with initial condition $f(\gamma(a)) = 0$. The estimated net change will be a sum of complex differentials, $\sum f'(z_k)\Delta z = \sum P(z_k)\Delta z$. The accumulation of these differentials can be visualised with mapping diagrams, so the sum estimates the line integral over the curve, $\int_\gamma P(z)dz$, the net change, $f(\gamma(b)) - f(\gamma(a))$, where f is a solution to the differential equation. The visualisations for the line integral can help see this accumulative limit, see figures 14 and 15.

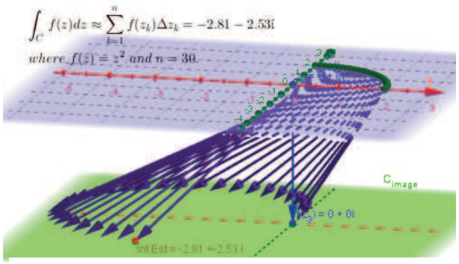


Figure 14. Mapping Diagram for Integral Estimate: $P(z)=z^2$ over $\frac{1}{4}$ circle $|z|=2$

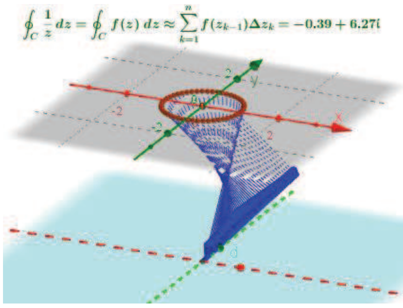


Figure 15.Mapping Diagram for Integral Estimate: $P(z)=\frac{1}{z}$ over $|z|=1$.

Not the End. Only a Beginning

This meagre summary cannot capture the power of dynamic figures to visualise the concepts covered in the sessions, nor can it show other concepts that mapping diagrams support. These can be found in the materials linked initially in this summary or by the reader in their own future explorations.