Introduction to Sensible Calculus: A Thematic Approach



The Anja S. Greer Conference on Mathematics, Science and Technology June 22 - June 27, 2014 Martin Flashman

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Day by Day Outline (Rev'd 6-26)

- 0. Sunday: Basic Themes Plus ...
 - Mapping Diagrams
 - Technology (Winplot and Geogebra)
- I. Monday: Making Sense of the Derivative.
- II. Tuesday: More on the Derivative
- III.Wednesday: DE's, Approximation and The Fundamental Theorem of Calculus
- IV.Thursday: More on the FT, DE's, Models and Estimations Making Sense of Taylor Theory
- V. Friday: Frontiers-Probability, Economics, Series, ...?

Continuing...

Newton's Method by Hand/Technology

- Using Spreadsheet: <u>Newton.xls</u>
- Using GeoGebra [New]: <u>Newton (ggb)</u>

Quick look at estimating important list of numbers: $\sqrt{2}, \pi, e, \ln(2)$

Euler's Method by Hand/Technology

- Using Spreadsheet: <u>Euler.xls</u>
- GeoGebra: Euler (ggb)
- Using Winplot: <u>diffeq.wp2</u>

Quick look at estimating important list of numbers: $\sqrt{2}, \pi, e, \ln(2)$

Euler's Method II y' = P(x, y)

•Euler's method also works for solving an initial value problem:

Given y' = P(x, y) and f(a) = c, find exactly or estimate f(b).

- One Step: the differential.
- Two Equal Steps: the differential reset after first step using y' = P(x, y).

-N Equal Steps: The differential reset after each step

Use of spread sheets to make the estimation systematic.

Euler's Method III
$$y'' = P(x, y, y')$$

•Euler's method also works for solving higher order initial value problems:

Given y'' = P(x, y, y') and f(a) = c, f'(a) = d, find exactly or estimate f(b).

- One Step: the differential.
- Two Equal Steps: the differential for y reset after first step using the differential for y'.

-N Equal Steps: The differential reset after each step

- Use of spread sheets to make the estimation systematic.

Partner Problems

One Problem per Partner pair. L.1 Assume y is a solution to the differential equation

 $\frac{dy}{dx} = \frac{1}{x^2 + 1}$ with y(0) = 2.

(a) Using just the given information, find any local extreme points for y and discuss the graph of y, including the issue of concavity. (b) Using the differential, estimate y(1) and y(-1).

L.2 Assume y is a solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{x^2 + 1}$$

(a) Sketch the tangent field showing tangents in all four quadrants.

- (b) Draw three integral curves on your sketch including one through the point (1, 2);
- (c) Suppose that a solution to the differential equation has value 2 at 1.
 - (i) Based on your graph, estimate the value of that solution at 2.

(ii) Estimate the value of y(3) using Euler's method with n = 4.

L.3 Assume y is a solution to the differential equation

$$\frac{dy}{dx} = -\frac{y}{x}$$

- (a) Sketch the tangent field showing tangents in all four quadrants.
- (b) Draw three integral curves on your sketch including one through the point (1, 2);
- (c) Suppose that a solution to the differential equation has value 2 at 1.
 - (i) Based on your graph, estimate the value of that solution at 2.
 - (ii) Estimate the value of y(2) using Euler's method with n = 4.

L.4 Suppose y'' = -y, y'(0) = 1 and y(0) = 0. Estimate y(1), y(2), y(3), and y(4).

The Sensible Calculus Program The Definite Integral, DE's, and Euler's Method The motivation for defining the definite integral comes from estimating a solution to an Initial Value Problem, visual and numerical estimation with graphs and mapping diagrams.

<u>V.A</u> The Definite Integral - Connecting the definition to Euler's method and DE's.

The consequences of this approach-

The FT of C makes sense.

Sensible Calculus: Two Forms of the Fundamental Theorem of Calculus

Evaluation Form

If f is continuous and G'(x) = f(x) for all x then

$$\int_a^b f(x) \, dx = G(b) - G(a).$$

Derivative Form (Barrow's Theorem) If f is continuous and $G(t) = \int_a^t f(x) dx$ then

G is a differentiable function and G'(t) = f(t).

Sensible Calculus: Two Forms of the Fundamental Theorem of Calculus

Derivative Form (Barrow's Theorem) If f is continuous and $G(t) = \int_{a}^{t} f(x) dx$ then

G is a differentiable function and G'(t) = f(t).

DE Interpretation: If f is continuous then there exists a solution to the DE: G'(t) = f(t).

The Fundamental Theorem of Calculus Derivative Form (Barrow's Theorem) If f is continuous and $G(t) = \int_a^t f(x) dx$ then G is a differentiable function and G'(t) = f(t). Interpretation:

f(x) is velocity of object at time x.

G(t) is the net change in position of object from time a to time t.

G'(t) = velocity of object at time t.

Sensible Calculus: Two Forms of the Fundamental Theorem of Calculus

Evaluation Form

If f is continuous and G'(x) = f(x) for all x then

$$\int_a^b f(x) \, dx = G(b) - G(a).$$

DE Interpretation: If f is continuous then the initial value problem: G'(t) = f(t) with G(a) = C has a unique solution:

$$G(\mathbf{t}) = \int_a^t f(x) \, dx + C.$$

Fundamental Theorem of Calculus Evaluation Form

If f is continuous and
$$G'(x) = f(x)$$
 for all x
then $\int_a^b f(x) dx = G(b) - G(a)$.

Interpretation:

G(x) is a position function for a moving object which has its velocity at time x given by f(x).

 $\int_{a}^{b} f(x) dx$ represents the net change in position of the object from time a to time b.

FT of Calculus Objective & Key Ideas

Two Objectives:

•Estimate Net Change in Distance from differential equation using Euler's method for a derivative function that depends only on x

•Measure the error in using Euler's method to estimate net change for monotonic functions. [To ensure existence of integral for continuous functions]

Sensible Proofs of FT (Derivative)

 Background properties of definite integral make sense (with geometry or motion) Assume *a* < *b*.

$$\int_{b}^{a} f = -\int_{a}^{b} f$$

If $m < f(x) < \text{for all } x \dots$ then
$$m(b-a) < \int_{a}^{b} f(x) \, dx < M(b-a)$$

$$\int_{a}^{b} f = \int_{a}^{c} f + \int_{c}^{b} f$$

Sensible Proofs of FT (Derivative)

 Background properties of definite integral make sense (with geometry or motion) Assume *a* < *b*.

If $G(t) = \int_a^t f(x) dx$ then use 4 steps to show G'(t) = f(t).

- Step I Interpretations: Geometry, Motion
- Step II Interpretations: Geometry, Motion
- Step III Interpretations: Geometry, Motion
- Step IV THINK!

FT of Calculus Objective & Key Ideas

Two Key Ideas:

•When x is close to a, f(x) is approximately equal to a linear function, f(a) + f'(a)(x-a).

•As long as f is a sufficiently well behaved function there is some c between a and x where

•
$$f(x) = f(a) + f'(c)(x-a)$$
.

Sensible Proofs of FT (Evaluation)

• Use Riemann Sum to estimate $\int_{a}^{b} f(x) dx$ then use Mean value theorem for each subinterval and telescoping sum!

Interpret with motion and geometry.

• Use FT of C Derivative to justify FT of C Evaluation.

Pedagogy: Fundamental Concepts, not "foundations".

We continue by considering approaches that use models and connect with the familiar.

Modeling contexts provide

- a sensible source for DE's
- practical and theoretical use of concepts and skills.

Pedagogical decision: Engage students in balanced approaches to models that are

- visual,
- symbolic,
- numeric, and
- verbal.

- <u>VI.A</u> Differential Equations & Models- The Exponential Function
- VI.B Differential Equations & Models- The Natural Logarithm Function
- VI.C Connecting the Natural Logarithm and Exponential Functions
- <u>VI.D</u> More Models & Inverse Trigonometry
- <u>IXA</u> Taylor Theory for e^x

- <u>VI.A</u> Differential Equations & Models- The Exponential Function Model Population:
- Biological cell growth
- Evolution of model from difference to differential equation with initial condition:
 - P(0) = 1

-
$$P(t + 1) - P(t) = P(t); \Delta P = P$$
: Solution: $P(t) = 2^t$

- dP = Pdt; $\frac{dP}{dt} = P$; Solution: ???
 - Direction Field
 - Euler's Method to estimate P(1) and P(t).
 - Exponential Properties evolve from the IVP. $P(t) = e^t$.
 - General solution for $\frac{dP}{dt} = k P$ with P(0) = A.

• VI.B Differential Equations & Models- The Natural Logarithm Function

Model Learning or slow growth Population:

- Evolution of model from heuristic on learning to differential equation with initial condition:
 - -L(1)=0
 - -L(t) > 0, L is increasing but at a decreasing rate due to fatigue. (Population growth but at a decreasing rate due to diminishing resources.)
 - $-\frac{dL}{dt}=\frac{1}{t}$; Solution: ???
 - - Direction Field
 - Euler's Method to estimate L(1) and L(t).
 - Connections to integration and FTof C
 - Logarithmic Properties evolve from the IVP. $L(t) = \ln(t)$.

• General solution for
$$\frac{dL}{dt} = \frac{k}{t}$$
 with $L(A) = 0$.

 VI.C Connecting the Natural Logarithm and Exponential Functions

- <u>VI.D</u> More Models & Inverse Trigonometry
- Other models for Learning and Poplulations

- Bounded growth:
$$L'(t) = \frac{1}{t^2}$$
; $L(1) = 0$.

$$-L'(t) = \frac{1}{t^2 + 1}; L(0) = 0.$$

Making Sense of Taylor Theory and the Calculus of Series

We continue by considering Taylor Theory through Modelling, Differential Equations, and Estimation.

What's Happening in The AP Calculus BC Course

- Applications: Series used for -Estimation of Numbers: $e, \pi, \sqrt{2}, ...$
 - -Estimation of Definite Integrals.
 - -Solution of **Differential Equations**.

What's Happening in The AP Calculus BC Course

• Critique

-Little motivation from previous work

-Newton's method.

-Estimates of Definite Integrals.

-Euler's Solutions to Differential Equations.

-Delayed Connection with the previous progress

-Unclear statement of what is fundamental.

Review: Sensible Calculus Approach

- Estimating the solution to differential equations such as y'' = -y or y'' = ywith y(0) = 1 and y'(0) = 1provides motivation for convergence questions of infinite series.
 - Euler's method using P(x, y, y') = -y or y also works for this!
 - P(x, y, y') = -y can be connected to a model for simple harmonic motion (a spring).
- The calculus of Taylor polynomials (not series) is a tool for approximating difficult definite integrals with a sensible control on the error.

Review: Sensible Calculus Approach

- From the beginning, the analysis of infinite sequences and series contains Taylor theory examples along with traditional examples of geometric and harmonic series.
- Historical connections: Newton's work in estimating the values of $\ln(2)$ and π illustrates how geometric series played a significant role in showing the power of the early calculus for computation and estimation.

Start with Connections

- Review of previous work on estimates that have been sequential.
- Estimates for integrals:
 - -constant,
 - -Linear,
 - -quadratic,

–interpolation (trapezoid, simpson) versus single point based on derivative (midpoint).

 Symbolic and computational ease of polynomials for derivatives and integrals. Illustrations from Sensible Calculus Taylor Theory and Series Taylor Theory without series. Motivating Example: Section IX.A introducing Taylor Theory for e[×] with applications. http://users.humboldt.edu/flashman/book/ch9/IXA. htm

Taylor Theory-Objective & Key Ideas

Objectives:

- Find estimating polynomials for a given function
- Measure the error in using polynomials to estimate.

Taylor Theory-Objective & Key Ideas

Key Ideas:

- When x is close to a, f(x) is approximately equal to a linear function, f(a) + f'(a)(x - a). [Linear Approximation-Differential]
- As long as f is a sufficiently well behaved function there is some c between a and x where

f(x) = f(a) + f'(c)(x - a). [MVT]

Focus Themes for Series: Estimation, Differential Equations, Models

IX.A. Focus on estimating a growth model with a differential equation: P'(x) = P(x), P(0) = 1.

- Solution is already treated
 - with estimation by Euler's method.
 - . "Exactly":

 $P(x) = e^x$

Focus Themes for Series: Estimation, Differential Equations, Models

IX.A. P'(x) = P(x), P(0) = 1.

- Estimation of the solution: Use the polynomial of degree n that best matches the differential equation.
- Determine estimate of error for estimating

• *e*

$$\cdot \int_0^1 e^{-x^2} dx$$

Discuss Sample Exercises from IX.A

- 1. Use the Taylor polynomial for e^x of degree 4 to estimate the following: $(a)e^2(b)e^3(c)e^{0.5}(d)e^{-1}(e)e^{3.14}$. [Spreadsheet helper supplied.]
- 2. Estimate e using the Taylor polynomial of degree n where n is (a) 6 (b) 7 (c) 8 (d) 10. In each estimate discuss the size of the error term R_n . [Spreadsheet helper.]
- 3. What value of n should be used so that the Taylor polynomial of degree n will give an estimate of e that is within .000001 of the exact value of e? Explain your result.
- 4. Use the Taylor polynomial for e^x of degree 5 to estimate $\int_0^1 e^{-t^2} dt$. Discuss the error in this approximation.

Sensible Calculus: EvolvingTaylorTheory

- . IX.A Taylor Theory for e^{\times}
- . IX.B MacLaurin Polynomials and Taylor Theory
- . IX.C MacLaurin Polynomials: How to Find Them
- . IX.D Taylor Polynomials

From Polynomials to Power Series

- · X.A Sequences
 - Examples
 - Definitions
 - Visualization
- X.B Series
 - Examples and Results: -Geometric Series
 - -Harmonic Series
 - -MacLaurin / Taylor Series
 - -Binomial Series

From Polynomials to Power Series

Tests for Divergence and Convergence.
Testing aimed at theme-power series and DE's from models.

-Key examples:

-Geometric Series

-Harmonic Series

-MacLaurin / Taylor Series

-Binomial Series

Power Series: A Thematic Sensible Conclusion

- . The Interval and Radius of Convergence.
- Functions and power series: $P(x) = \sum_{0}^{\infty} a_n x^n$.
 - . Is P a differentiable function?
 - What is the derivative of P?
 - . Is $P \ a \ C^{\infty}$ function?
 - What are the Taylor polynomials and the Taylor series for P?
- Discussion: What is the relation of functions defined by power series to solving differential equations?

Conclusion for Thursday.

With this reorganization, the treatment of sequences and series forms a sensible part of the first year calculus program, capping an approach that focuses on understanding the three mathematical themes:

- . Differential Equations,
 - . Estimation, and
- . Mathematical Modeling.

End of Session IV

• Questions?

The End!

Questions for next session? Catch me between sessions or e-mail them to me: <u>flashman@humboldt.edu</u>