Introduction to Sensible Calculus: A Thematic Approach

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Day by Day Outline (Rev'd 6-25)

0. Sunday: Basic Themes Plus …
   - Mapping Diagrams
   - Technology (Winplot and Geogebra)

I. Monday: Making Sense of the Derivative.

II. Tuesday: More on the Derivative

III. Wednesday: DE's, Approximation and The Fundamental Theorem of Calculus

IV. Thursday: More on DE's, Models and Estimations. Making Sense of Taylor Theory and the Calculus of Series.

V. Friday: Frontiers-Probability, Economics, …

Daily Assignment
Submit on paper or electronically.

- **Create one exercise and one problem** that incorporates (and/or extends) something from the session content.
- **Pose one question** related to the class content that you would like explained further. [I will respond privately unless you grant permission for a public response.]
- Take one (or two) topics discussed in the session and **discuss how you can incorporate** its content or technology into your teaching.
- **Electronic submissions may be shared with the class through the course webpage with submitter's permission.**
- **OPTIONAL: Complete any worksheet or problems suggested during class.**

Continuing from Last Class

Review GeoGebra
Solution to Continuity Problem

- **Download: Cont Prob 2 (ggb)**

Making Sense of the Calculus of Derivatives

- Finding derivatives from the definition can be tedious for more complicated elementary functions.
- The calculus is a systematic procedure for finding the derivatives of elementary functions.
- An elementary function is a function built from a list of core functions by applying addition, subtraction, multiplication, division, and composition to the core functions and their inverses.
- The Core Functions (Short list): $c, x^n, e^x, \sin(x)$
- (Others) $x^r,\ b^x,\ \ln(x), \cos(x), \tan(x), \sec(x)$
- Rules: Linearity, Product, Quotient, Chain

Making Sense of a Differential Equation and the Fundamental Theorem of Calculus

- Example: The following differential equations of the form $\frac{dy}{dx} = P(x)$ have solutions that cannot be expressed as an elementary function.
  - $\frac{dy}{dx} = \sin(x^2)$
  - $\frac{dy}{dx} = e^{-x^2}$
- The solutions to these are given by using the FT of C:

$$y = f(t) = \int_0^t P(x)dx$$

The Fundamental Theorem of Calculus says:

When $P(x)$ is continuous, then $\frac{dy}{dt} = P(t)$.

The Fundamental Theorem of Calculus Derivative Form

If $\boldsymbol{f}$ is continuous and $\boldsymbol{G(t) = \int_a^t f(x)\,dx}$ then

$\boldsymbol{G}$ is a differentiable function and $\boldsymbol{G'(t) = f(t)}$.

Interpretation:

$f(x)$ is velocity of object at time x.

$G(t)$ is the net change in position of object from time a to time t.

$G'(t)$ = velocity of object at time t.

Making Sense of Calculus: Applications to Estimation

- Intermediate Value Theorem, Roots and Continuity. SC I.I.2. Intermediate Values
  - Bisection Algorithm
    - Graphical
    - Mapping Diagrams
  - Spreadsheets

Making Sense of Calculus: Applications to Estimation

- Linearity and Estimating Roots
  III.A.2
  - Linear Estimation Function:
    - Geometric Interpretation (Slope of Tangent line)
    - Motion Interpretation (Mapping Diagram, Magnification and Focus Point)
  - Solving for roots in linear functions.
    - Brief excursion into inverses for linear functions.
    - More mapping diagrams!
  - Newton's Method Algorithms. Estimation appications to error estimates.

Examples on Excel, Winplot, Geogebra

- **Excel example(s):**
  - Linear Mapping Diagram example
  - Newtons Method
- **Winplot examples:**
  - **Linear Mapping Diagram-composition examples**
  - **Linear Graph Linked File-composition examples**
- Geogebra examples:
  - IV Steps
  - Secant Tangent
  - Alternative Derivative for Sine.

Session III Differential Equations, Approximation and The Fundamental Theorem of Calculus

We continue to explore making calculus sensible by a consideration of the **FT of Calculus from a view of DE's and estimations using Euler's Method interpreted in a variety of contexts.**

Review: What's Happening Now in The First Calculus Course

- Differential Calculus: The derivative and applications- graphing, extremes, rates, Newton's method, mixing continuity and differentiability in theory, some slight mention of differential equations, THEN...
- Integral Calculus! Area, area, area, then Magic!
- The Fundamental Theorems of Calculus

What's Happening Now

Critique:

- **Little motivation for integration from previous work despite**
    - **Local analysis of functions based on the derivative and MVT.**
    - **Estimation connected to**
        - **the derivative definition**
        - **linear approximating function (tangent line interpretation)**
        - **The differential**
    - **Introduction to Differential Equations available through implicit differentiation and related rates**
- **Unclear: What are the fundamental mathematical questions for a model?**

Sensible Calculus: Make Connections

- Related Rates and Implicit differentiation involve "differential equations"
- Work on graphing using the derivative involves making qualitative inferences about a function from information about its derivative.
- Applications of the Mean Value Theorem suggest uniqueness of solution to IVP.
- Estimates using the differential (linear estimator).

Sensible Calculus: Two Forms of the Fundamental Theorem of Calculus

Evaluation Form

If f is continuous and $G'(x) = f(x)$ for all x .... then

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Derivative Form (Barrow's Theorem)

If f is continuous and $G(t) = \int_a^t f(x)\,dx$ then

G is a differentiable function and $G'(t) = f(t)$.

Fundamental Theorem of Calculus
Evaluation Form

If f is continuous and $\boldsymbol{G'(x)} = \boldsymbol{f(x)}$ for all x ....
then $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f(x)\,dx} = \boldsymbol{G(b)} - \boldsymbol{G(a)}$.

Interpretation:

$G(x)$ is a position function for a moving object which has its velocity at time x given by $f(x)$.

$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f(x)\,dx}$ represents the net change in position of the object from time a to time b.

The Fundamental Theorem of Calculus
Derivative Form (Barrow's Theorem)

If $\boldsymbol{f}$ is continuous and $\boldsymbol{G(t) = \int_a^t f(x)\,dx}$ then $\boldsymbol{G}$ is a differentiable function and $\boldsymbol{G'(t) = f(t)}$.

Interpretation:

$f(x)$ is velocity of object at time x.

$G(t)$ is the net change in position of object from time a to time t.

$G'(t)$ = velocity of object at time t.

The **FT of Calculus, DE's, and Euler's Method**

The motivation for the FT of C comes from estimating a solution to an Initial Value Problem: visual and numerical estimation with graphs and mapping diagrams.

Ch III.A.1. THE DIFFERENTIAL

Ch IV Differential Equations from an Elementary Viewpoint

V.A The Definite Integral – Connecting the definition to Euler's method and DE's.

Visualizing solutions to IVP's

Initial Value Problem (IVP) :

Given $y' = f'(x) = P(x)$ *or* $P(x, y)$ **and** $f(a) = c$**, find exactly or estimate** $f(b)$.

Connect the differential equation to the geometric interpretation using direction (tangent) fields and integral curves. Visual estimate of solution. See **Sensible Calculus** **on DE's.**

Estimating solutions to IVP's

Initial Value Problem (IVP) :

Given $y' = f'(x) = P(x)$ **and** $f(a) = c$**,**

find exactly or estimate $f(b)$.

Connect to previous work on estimates using the differential (linear estimator).

Euler's method evolves from a progression of estimates for solving an initial value problem.

Euler's Method

- Euler's method evolves from a progression of estimates for solving an initial value problem:

Given $y' = f'(x) = P(x)$ **and** $f(a) = c$**, find exactly or estimate** $f(b)$.

  - One Step: the differential.
  - Two Equal Steps: the differential reset after first step.

- N Equal Steps: The differential reset after each step

  - Use of spread sheets to make the estimation systematic.

- **Ease of estimation of net change when f '(x) depends only on x.**

Euler's Method by Hand/Technology

- Using Spreadsheet and/or GeoGebra
  - **Euler.xls**
  - **Euler (ggb)**
- Using Winplot
  - **diffeq.wp2**

Euler's Method II
$y' = P(x, y)$

- Euler's method also works for solving an initial value problem:

Given $y' = P(x, y)$ **and** $f(a) = c$**, find exactly or estimate** $f(b)$.

  - One Step: the differential.
  - Two Equal Steps: the differential reset after first step using $y' = P(x, y)$.

– N Equal Steps: The differential reset after each step

  - Use of spread sheets to make the estimation systematic.

Euler's Method III
$y'' = P(x, y, y')$

- Euler's method also works for solving higher order initial value problems:

Given $y'' = P(x, y, y')$ and $f(a) = c, f'(a) = d$, find exactly or estimate $f(b)$.

  - One Step: the differential.
  - Two Equal Steps: the differential for y reset after first step using the differential for y'.

- N Equal Steps: The differential reset after each step

  - Use of spread sheets to make the estimation systematic.

Partner Problems
One Problem per Partner pair.

L.1 **Assume y is a solution to the differential equation**

$$\frac{dy}{dx} = \frac{1}{x^2+1} \text{ with } y(0) = 2.$$

(a) Using just the given information, find any local extreme points for y and discuss the graph of y, including the issue of concavity.
(b) Using the differential, estimate $y(1)$ and $y(-1)$.

L.2 Assume y is a solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{x^2+1}$$

(a) Sketch the tangent field showing tangents in all four quadrants.
(b) Draw three integral curves on your sketch **including one through the point** $(\mathbf{1}, \mathbf{2})$;
(c) Suppose that a solution to the differential equation has value 2 at 1.
  (i) Based on your graph, **estimate** the value of that solution at 2.
  (ii) Estimate the value of $y(3)$ using Euler's method with $n = 4$.

L.3 **Assume y is a solution to the differential equation**

$$\frac{dy}{dx} = -\frac{y}{x}$$

(a) Sketch the tangent field showing tangents in all four quadrants.
(b) Draw three integral curves on your sketch **including one through the point** $(\mathbf{1}, \mathbf{2})$;
(c) Suppose that a solution to the differential equation has value 2 at 1.
  (i) Based on your graph, **estimate** the value of that solution at 2.
  (ii) Estimate the value of $y(2)$ using Euler's method with $n = 4$.

L.4 Suppose $y'' = -y$, $y'(0) = 1$ and $y(0) = 0$. Estimate $y(1), y(2), y(3)$, and $y(4)$.

The Sensible Calculus Program

The Definite Integral, DE's, and Euler's Method

The motivation for defining the definite integral comes from estimating a solution to an Initial Value Problem, visual and numerical estimation with graphs and mapping diagrams.

V.A The Definite Integral – Connecting the definition to Euler's method and DE's.

The consequences of this approach-

The FT of C makes sense.

FT of Calculus
Objective & Key Ideas

Two Objectives:

- Estimate Net Change in Distance from differential equation using Euler's method for a derivative function that depends only on x
- Measure the error in using Euler's method to estimate net change for monotonic functions.

FT of Calculus
Objective & Key Ideas

Two Key Ideas:

- When x is close to a, f(x) is approximately equal to a linear function, $f(a) + f'(a)(x-a)$.
- As long as f is a sufficiently well behaved function there is some c between a and x where

  - $f(x) = f(a) + f'(c)(x-a)$.

Conclusion

- With this reorganization, the treatment of the Fundamental Theorem of Calculus forms a sensible part of the first year calculus program, in a thematic approach to understanding the mathematical themes:

- **Differential Equations,**
- **Estimation, and**
- **Mathematical Modeling**

Examples on Excel, Winplot, Geogebra

- **Excel example(s):**
  - Euler's Method
- **Winplot examples:**
  - **Linear Mapping Diagram-composition examples**
  - **Linear Graph Linked File-composition examples**
- Geogebra examples:
  - Euler's Method with Mapping diagram & rectangles

End of Session III

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Questions for next session?
Catch me between sessions or e-mail them to me:
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- [FL1] Flashman, Martin. "Differential Equations: A Motivating Theme for A Sensible Calculus," in "Calculus for All Users" The Report of A Conference on Calculus and Its Applications Held at the University of Texas, San Antonio, NSF Calculus Reform Conference, October 5 - 8, 1990.
- [UMAP] Flashman, Martin. "*A Sensible Calculus.*," The UMAP Journal, Vol. 11, No. 2, Summer, 1990, pp. 93-96.
- [FL2] Flashman, Martin. "Using Computers to Make Integration More Visual with Tangent Fields," appearing in Proceedings of the Second Annual Conference on Technology in Collegiate Mathematics, Teaching and Learning with Technology of November 2-4, 1989, edited by Demana, Waits, and Harvey, Addison-Wesley, 1991.
- [FL3] Flashman, Martin. "Concepts to Drive Technology," in Proceedings of the Fifth Annual Conference on Technology in Collegiate Mathematics, November 12-15, 1992, edited by Lewis Lum, Addison-Wesley, 1994.
- [FL4] Flashman, Martin. "Historical Motivation for a Calculus Course: Barrow's Theorem," in Vita Mathematica: Historical Research and Integration with Teaching, edited by Ronald Calinger, MAA Notes, No. 40, 1996.