Introduction to Sensible Calculus: A Thematic Approach

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Day by Day Outline

0. Sunday: Basic Themes Plus …
   - Mapping Diagrams
   - Technology (Winplot and Geogebra)

I. Monday: Making Sense of the Derivative.

II. Tuesday: DE's, Approximation and The Fundamental Theorem of Calculus

III. Wednesday: More on DE's, Models and Estimations

IV. Thursday: Making Sense of Taylor Theory and the Calculus of Series

V. Friday: Frontiers-Probability, Economics, …

Daily Assignment
Submit on paper or electronically.

- **Create one exercise and one problem** that incorporates (and/or extends) something from the session content.
- **Pose one question** related to the class content that you would like explained further. [I will respond privately unless you grant permission for a public response.]
- Take one (or two) topics discussed in the session and **discuss how you can incorporate** its content or technology into your teaching.
- **Electronic submissions may be shared with the class through the course webpage with submitter's permission.**
- **OPTIONAL: Complete any worksheet or problems suggested during class.**

Continuing from Last Class

Sensible Calculus and Mapping Diagram Resources

- **Sensible Calculus Visualizations and Mapping Diagrams**
- **Mapping Diagrams from A(lgebra) B(asics) to C(alculus) and D(ifferential) E(quation)s.**
  A Reference and Resource Book on Function Visualizations Using Mapping Diagrams

Examples withTechnology

LINK for Current Materials

- **Excel examples [Covered last class]**
- **Winplot examples**
- **Geogebra examples**
- **SketchPad examples**

Simple Examples are important!

$\boldsymbol{f(x) = mx + b}$ with a mapping diagram –

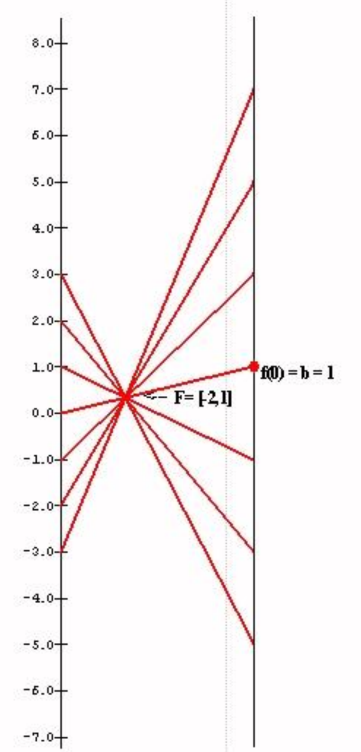
Five examples:

- Example 1: $m = -2;\ b = 1$: $f(x) = -2x + 1$
- Example 2: $m = 2;\ b = 1$: $f(x) = 2x + 1$
- Example 3: $m = ½;\ b = 1$: $f(x) = ½\,x + 1$
- Example 4: $m = 0;\ b = 1$: $f(x) = 0\,x + 1$
- Example 5: $m = 1;\ b = 1$: $f(x) = x + 1$

Visualizing $f(x) = \boldsymbol{m}x + \boldsymbol{b}$ with a mapping diagram -- Five examples:

Example 1: $\boldsymbol{m} = -2;\ \boldsymbol{b} = 1$

$f(x) = -2x + 1$



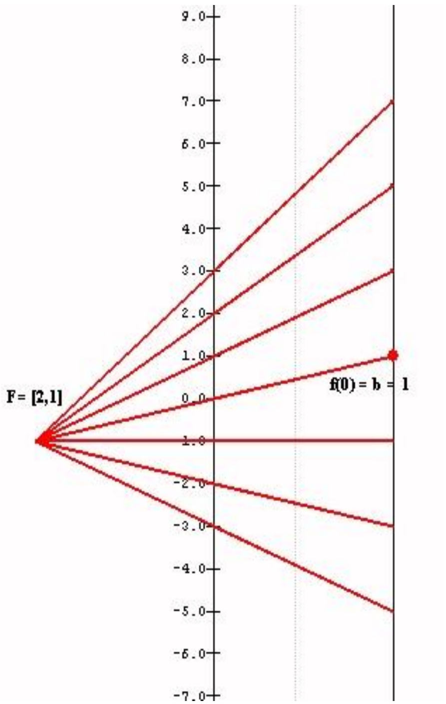
- Each arrow passes through a single point, which is labeled $\boldsymbol{F} = [-2, 1]$.
- The point $\boldsymbol{F}$ completely determines the function f.
  - **given a point** / number, x, on the source line,
  - there is a **unique arrow passing through** $\boldsymbol{F}$
  - **meeting** the target line at **a unique point** / number, $-2x + 1$,

which corresponds to the linear function's value for the point/number, x.

Visualizing $f(x) = \boldsymbol{m}x + \boldsymbol{b}$ with a mapping diagram -- Five examples:

Example 2: $\boldsymbol{m} = \mathbf{2}$; $\boldsymbol{b} = \mathbf{1}$

$f(x) = \mathbf{2}x + \mathbf{1}$



- Each arrow passes through a single point, which is labeled $\boldsymbol{F} = [\mathbf{2}, \mathbf{1}]$.
- The point $\boldsymbol{F}$ completely determines the function f.
  - **given a point** / number, x, on the source line,
  - there is a **unique arrow passing through $\boldsymbol{F}$**
  - **meeting** the target line at **a unique point** / number, $2x + 1$,

  which corresponds to the linear function's value for the point/number, x.

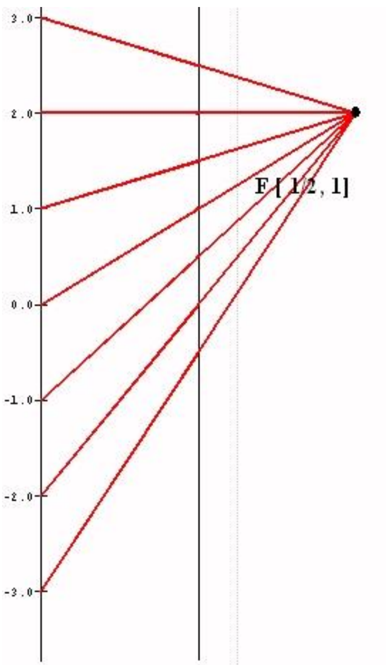
Visualizing $f(x) = \boldsymbol{m}x + \boldsymbol{b}$ with a mapping figure -- Five examples:

Example 3: $\boldsymbol{m} = \frac{1}{2}$; $\boldsymbol{b} = \mathbf{1}$

$$\boldsymbol{f(x) = \frac{1}{2}x + 1}$$

- Each arrow passes through a single point, which is labeled $\boldsymbol{F} = \left[\frac{1}{2}, \mathbf{1}\right]$.
- The point $\boldsymbol{F}$ completely determines the function f.
  - **given a point** / number, x, on the source line,
  - there is a **unique arrow passing through $\boldsymbol{F}$**
  - **meeting** the target line at **a unique point** / number, $\frac{1}{2}x + 1$,

  which corresponds to the linear function's value for the point/number, x.



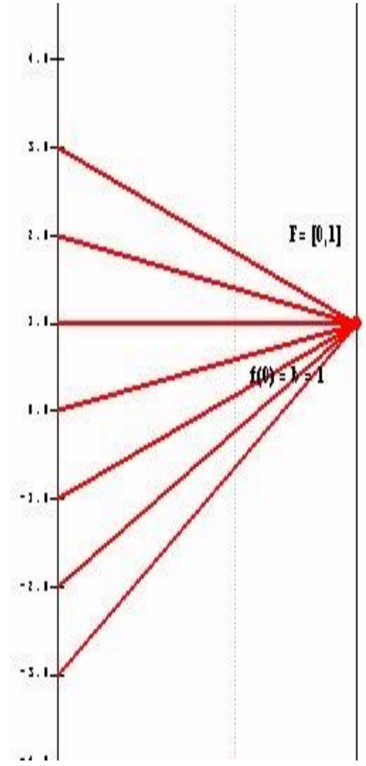
Visualizing $f(x) = \boldsymbol{m}x + \boldsymbol{b}$ with a mapping figure -- Five examples:

Example 4: $\boldsymbol{m} = \mathbf{0}$; $\boldsymbol{b} = \mathbf{1}$

$$f(x) = \mathbf{0}x + \mathbf{1}$$

- Each arrow passes through a single point, which is labeled $\boldsymbol{F} = [\mathbf{0}, \mathbf{1}]$.
- The point $\boldsymbol{F}$ completely determines the function f.
  - **given a point** / number, x, on the source line,
  - there is a **unique arrow passing through** $\boldsymbol{F}$
  - **meeting** the target line at **a unique point** / number, $0x + 1$,

  which corresponds to the linear function's value for the point/number, x.

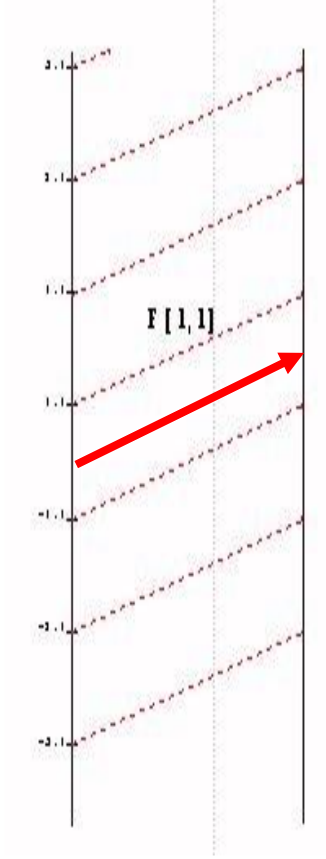


Visualizing $f(x) = \boldsymbol{m}x + \boldsymbol{b}$ with a mapping figure -- Five examples:

Example 5: $\boldsymbol{m} = \mathbf{1}$; $\boldsymbol{b} = \mathbf{1}$

$f(x) = 1x + \mathbf{1}$

- Unlike the previous examples, in this case it is not a single point that determines the mapping figure, but the single arrow from 0 to 1, which we designate as $F[1,1]$
- It can also be shown that this single arrow completely determines the function. Thus, given a point / number, x, on the source line, there is a unique arrow passing through $\boldsymbol{x}$ ***parallel to*** $F[1,1]$ meeting the target line a unique point / number, $x + 1$, which corresponds to the linear function's value for the point/number, x.
  - The single arrow completely determines the function f.
    - **given a point** / number, x, on the source line,
    - there is a **unique arrow** through $\boldsymbol{x}$ ***parallel to*** $F[1,1]$
    - **meeting** the target line at **a unique point** / number, $x + 1$, which corresponds to the linear function's value for the point/number, x.



I Making Sense of Calculus: The Derivative.

Motivation from Models
Balance in Interpretations
Exact vs. Estimated
Local Linearity
Unification/Generalization
Abstraction/Application (D. Solow)

Session I Making Sense of the Derivative.

Reconsideration of the linear function

$$y = f(x) = mx + b$$

Interpretation Model context	x	$y = f(x)$	m
Motion	(t) time	(s) position on a coordinate line	Velocity of object (Constant)
Coordinate Geometry	1st coordinate of a point in the plane	2nd coordinate of a point in the plane	Slope of line
Economic Production	quantity of a product bought/sold	total cost/revenue/profit for product bought/sold	marginal cost/revenue /profit of product (Constant)
Probability Random Variable	range value of a given random variable	probability that the random variable will be less than or equal to the given range	probability density of random variable

Making Sense of Calculus: The Derivative.

- Ch 0 A Motivation: What is the calculus?
- Ch 0 B Solving the Tangent Problem

The derivative as a number:

- a modelling context
- visual and numerical estimation
  - graphs
  - mapping diagrams.
- Ch 1 A Tangent Line
- Ch 1 B Velocity

Making Sense of Calculus: The Derivative.

- Ch 0 A Motivation: What is the calculus?

Making Sense of Calculus: The Derivative.

- Ch 0 A Motivation: What is the calculus?

The derivative as a number:

- a modelling context
- visual and numerical estimation
  - graphs (Slope)
  - mapping diagrams (Magnification)

Making Sense of Calculus: The Derivative.

- Ch 0 B Solving the Tangent Problem

The derivative as a number:

- a modelling context
- visual and numerical estimation
  - graphs (Slope)

Making Sense of Calculus: The Derivative.

The derivative as a number:

- a modelling context
- visual and numerical estimation
  - graphs (Slope)
- Ch 1 A Tangent Line
  - Estimation
  - Four Steps
- **UNIFICATION /GENERALIZATION**

Making Sense of Calculus: The Derivative.

The derivative as a number:

- a modelling context
- visual and numerical estimation
  - mapping diagrams (Magnification)
- Ch 1 B Velocity
  - Estimation
  - Four Steps
- **UNIFICATION /GENERALIZATION**

Making Sense of Calculus: The Derivative.

The derivative as a number:

- **ABSTRACTION**
- visual and numerical estimation
  - graphs (Slope)
  - mapping diagrams (Magnification)
- Ch 1 D Derivative
  - Estimation
  - Four Steps

Making Sense of Calculus: The Derivative.

The derivative as a number:

- a modelling context: Generalization/Abstraction/Application
- visual and numerical estimation
  - graphs (Slope)
  - mapping diagrams (Magnification)
- Ch 1 C Other Models/Interpretations for the Derivative
  - Probability: (Point) Probability Density
  - Economics: Marginal Cost/Revenue/Profits

Think about These Problems

M.1 Use a **mapping figure for the function** $f(x) = -3x + 2$ **to illustrate that**

$$f'(1) = -3.$$

Sketch a mapping figure that illustrates the work to show that **the linear function** $f(x) = mx + b$ has $f'(a) = m$. **Discuss how different values of** m **impact your figure.**

M.2 Use a **mapping figure for the function** $f(x) = x^2$ **to illustrate that**

$$f'(3) = 6.$$

Sketch a mapping figure that illustrates the work to show that $f'(a) = 2a$.

M.3 Use a **mapping figure for the function** $f(x) = \frac{1}{x}$ **to illustrate that**

$$f'(2) = -\frac{1}{4}.$$

Sketch a mapping figure that illustrates the work to show that $f'(a) = -\frac{1}{a^2}$.

Making Sense of Calculus: The Derivative of Core Functions

- The Exponential Function
  SC I.F.2
- Motivate with Population Model:

$$P(t) = 2^t$$

  - Find $P'(0)$.
  - Connect $P'(t) = P'(0)\,P(t)$
- Connect to Compound Interest Rate
- Interpretation and definition of e.
- If $P'(x) = P(x)$ with $P(0) = 1$ then $P(x) = e^x$.

Making Sense of Calculus: The Derivative of Core Functions

- *The Sine Function*

  [*I.F.3*]

- *Motivate with Unit Circle Motion*:

$$f(t) = \sin(t), g(t) = \cos(t)$$

  - $Estimate, then\ find\ f'(0), g'(0).$
  - $Connect\ f'(t) = f'(t)g(0) + g'(t)f(0)$

Making Sense of Calculus: The Derivative Calculus

- Product Rule SC II.A
- Motivate with Linearity in Algebra
  - Linear Estimation
- Connect to Rate Interpretation
  - Rectangular Area
  - Mapping Diagram of Sides
  - Error Estimation
- Continuity and Differentiability Connection

Making Sense of Calculus: The Derivative Calculus

- Chain Rule SC II.B
- Motivate with Linearity in Algebra
  - Linear Estimation
- Connect to Rate Interpretation
  - Gas consumption, Motion, Time
  - Mapping Diagram for Composition
  - Error Estimation
  - Pattern Recognition
  - Leibnitz Notation

Making Sense of Calculus: Applications to Estimation

- Local Linearity and the Differential III.A.1
  - Linear Estimation Function:
    - Geometric Interpretation (Slope of Tangent line)
    - Motion Interpretation (Mapping Diagram, Magnification and Focus Point)
  - Leibniz Notation and the Differential
  - Estimation applications to error estimates.

Making Sense of Calculus: Applications to Estimation

- Intermediate Value Theorem, Roots and Continuity.
  SC I.I.2. Intermediate Values
  - Bisection Algorithm
    - Graphical
    - Mapping Diagrams
  - Spreadsheets

Making Sense of Calculus: Applications to Estimation

- Linearity and Estimating Roots
  III.A.2
  - Linear Estimation Function:
    - Geometric Interpretation (Slope of Tangent line)
    - Motion Interpretation (Mapping Diagram, Magnification and Focus Point)
  - Solving for roots in linear functions.
    - Brief excursion into inverses for linear functions.
    - More mapping diagrams!
  - Newton's Method Algorithms, Estimation

Examples on Excel, Winplot, Geogebra

- **Excel example(s):**
  - Linear Mapping Diagram example
  - Newtons Method
- **Winplot examples:**
  - **Linear Mapping Diagram-composition examples**
  - **Linear Graph Linked File-composition examples**
- Geogebra examples:
  - IV Steps
  - Secant Tangent
  - Alternative Derivative for Sine.

End of Session I

- Next session will deal further with inferential applications of the derivative as a transition to the study of differential equations, integration and the Fundamental Theorem of Calculus
- Questions?

Thanks

The End!

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Questions for next session?

Catch me between sessions or e-mail them to me:

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- [FL1] Flashman, Martin. "Differential Equations: A Motivating Theme for A Sensible Calculus," in "Calculus for All Users" The Report of A Conference on Calculus and Its Applications Held at the University of Texas, San Antonio, NSF Calculus Reform Conference, October 5 - 8, 1990.
- [UMAP] Flashman, Martin. "*A Sensible Calculus.*," The UMAP Journal, Vol. 11, No. 2, Summer, 1990, pp. 93-96.
- [FL2] Flashman, Martin. "Using Computers to Make Integration More Visual with Tangent Fields," appearing in Proceedings of the Second Annual Conference on Technology in Collegiate Mathematics, Teaching and Learning with Technology of November 2-4, 1989, edited by Demana, Waits, and Harvey, Addison-Wesley, 1991.
- [FL3] Flashman, Martin. "Concepts to Drive Technology," in Proceedings of the Fifth Annual Conference on Technology in Collegiate Mathematics, November 12-15, 1992, edited by Lewis Lum, Addison-Wesley, 1994.
- [FL4] Flashman, Martin. "Historical Motivation for a Calculus Course: Barrow's Theorem," in Vita Mathematica: Historical Research and Integration with Teaching, edited by Ronald Calinger, MAA Notes, No. 40, 1996.
- [DS1] Solow, Daniel. *The Keys to Advanced Mathematics : Recurrent Themes in Abstract Reasoning*
- [DS2] Solow, Daniel. **How to Read and Do Proofs: An Introduction to Mathematical Thought Processes,**Wiley; 6 ed (July 29, 2013).