**Part I: Core Problems** (Do these without using summary sheets)

1. (10 Points) Suppose $y = (5t^2 + \sin(t))^4$ . Find $\frac{dy}{dt}$ .

**Solution:** Use the chain rule- $y = (5t^2 + \sin(t))^4 = u^4$; $u = 5t^2 + \sin(t)$ So

Ans. $dy/dt = 4(5t^2 + \sin(t))^3 \cdot (10t + \cos(t))$

2. (10 Points) Use calculus to **find the maximum value** of $g(x) = 5x^3 - 15x + 1$ for $x$ in the interval [-2,2].

**Show all your work.**

**Solution:** $g'(x) = 15x^2 - 15 = 15(x+1)(x-1)$ .

So the critical numbers where $g'(x) = 0$ in the interval [-2,2] are $x = +1$ and $x = -1$.

| $x$ | $g(x)$ |
|---|---|
| 2 | 5*8-15*2+1 = 11 |
| 1 | 5-15+1= -9 |
| -1 | -5+15+1= 11 |
| -2 | 5*(-8)+15*2 +1 = -9 |

Ans. The maximum value for g($x$) for $x$ in the interval [-2,2] is 11.

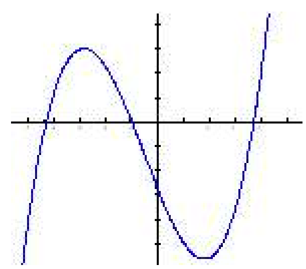
3.( 10 Points) Suppose $f$ is a function with $f'(x) = 6x + 3$ and $f(1) = 2$ . Find $f(0)$.

**Solution:** $f(x) = 3x^2 + 3x + C$. $f(1) = 2 = 3 + 3 + C$ so C = 2-6= -4.
Thus $f(0) = -4$.
Ans. $f(0) = -4$.

4. (10 points) Suppose the graph of $y = P(x)$ is given below. Based on this graph, estimate the interval(s) where

a. $P'(x) < 0$: Estimate: [-3, 2]

b. $P''(x) > 0$ : Estimate: [-1, 4.3]



Graph of $y = P(x)$

5.( 10 points) . **For this problem suppose that P is a continuous function with P(1) = 4, P(1.5) = -6, P(2) = 2, and P(2.5) = 8.**
Suppose that **G'($x$) = P($x$)** for all $x$ and **G(1)= 2**. Use Euler's method with N = 4 to **estimate G(3).**

**Solution:** dx = (3-1)/4 = 1/2.

| $x$ | **G($x$)** | **G'($x$) = P($x$)** | **dG = P ($x$)** d$x$ |
|---|---|---|---|
| 1 | 2 | 4 | 2 |
| 1.5 | 4 | -6 | -3 |
| 2 | 1 | 2 | 1 |
| 2.5 | 2 | 8 | 4 |
| 3 | 6 | | |

Ans. G(3) ≈ 6

**Part II: Extended Problems (Done with use of summary sheets)**

**Solutions**

6. (10 points)
a. Complete the following **DEFINITION** for the derivative:
The **derivative** of the function $f$ at $a$, denoted $f'(a)$, is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ provided the limit exists.

b. **USING THE DERIVATIVE DEFINITION,** find $f'(1)$ when $f(x) = 3x^2 + 5x$.
**Solution:**

Step 1: $f(1+h) = 3(1+h)^2 + 5(1+h) = 3 + 6h + 3h^2 + 5 + 5h$
$- f(1) = 3 + 5$

Step 2: $f(1+h) - f(1) = 11h + 3h^2$

Step 3: $\dfrac{f(1+h) - f(1)}{h} = \dfrac{11h + 3h^2}{h} = 11 + 3h$

Step 4: As $h \to 0$, $\dfrac{f(1+h) - f(1)}{h} = 11 + 3h \to 11$, so $f'(1) = 11$.

Ans. $f'(1) = 11$

7.(10 points) **Complete the following as most appropriate**:
a. **Definition:** The function $f$ is **continuous** at ($x =$ ) $a$ if
(i) $f(a)$ exists; (ii) $\lim_{x \to a} f(x)$ exists; and (iii) $\lim_{x \to a} f(x) = f(a)$

b. **The Intermediate Value Theorem:**
If $f$ is a continuous function on the interval $[a,b]$ and Z is a number between $f(a)$ and $f(b)$ then **there is a number *p*** where $a < p < b$ **and** $f(p) = Z$.

**c. The Mean Value Theorem**:
If $f$ is a continuous function on the interval $[a,b]$ and also a differentiable function on the interval $(a,b)$ then **there is a number *p*** where $a < p < b$ **and**

$$f'(p) = \frac{f(b) - f(a)}{b - a}$$

8.(20 points) Find the following derivatives as indicated:

a. $y = 3x^4 - \sqrt{x} - \dfrac{3}{x^4}$. Find dy/d**x**. Solution: $\dfrac{dy}{dx} = 12x^3 - \dfrac{1}{2\sqrt{x}} + \dfrac{12}{x^5}$

b. $f(x) = \dfrac{\cos(x)}{x^2 + 4}$ . Find $f'(x)$.

Solution: $f'(x) = \dfrac{-(x^2+4)\sin(x) - 2x\cos(x)}{(x^2+4)^2}$

c. Z = sec(3+5t). Find dZ/dt. Solution: dZ/dt = 5 sec(3+5t) tan (3+5t)

d. $z = 3x\sin(y) - y e^x - x^2 \ln(y)$ . Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ .

Ans. $\dfrac{\partial z}{\partial x} = 3\sin(y) - y e^x - 2x\ln(y)$ ; $\dfrac{\partial z}{\partial y} = 3x\cos(y) - e^x - \dfrac{x^2}{y}$

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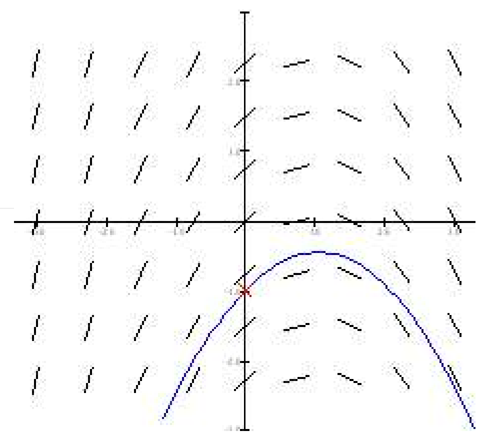
9. (10 points) Sketch the tangent (direction) field for **dy/dx =1 - x** in all quadrants with **-2≤ x ≤2.** **Draw an integral curve for this equation on your graph illustrating the graph of a solution to this differential equation passing through the point (0, -1).**



10. (20 points) Find the following **limits** (when possible): [Show your work!]

a. $\lim_{x\to\infty} \dfrac{4x^2 - 200x + 1000}{8x^2 + 200x - 1000} =$ 1/2

Solution:

$$\lim_{x\to\infty} \frac{4x^2 - 200x + 1000}{8x^2 + 200x - 1000} = \lim_{x\to\infty} \frac{(4x^2 - 200x + 1000)\frac{1}{x^2}}{(8x^2 + 200x - 1000)\frac{1}{x^2}} = \lim_{x\to\infty} \frac{4 - \frac{200}{x} + \frac{1000}{x^2}}{8 + \frac{200}{x} - \frac{1000}{x^2}} = \frac{1}{2}$$

b. $\lim_{x\to 2^+} \frac{x^2-5x+6}{x-2} = $ __-1__

Solution: $\lim_{x\to 2^+} \frac{x^2-5x+6}{x-2} = \lim_{x\to 2^+} \frac{(x-3)(x-2)}{x-2} = \lim_{x\to 2^+} x-3 = -1$

c. $\lim_{x\to 0} \frac{\sin(x) - x}{e^x - x - 1} =$ __________

Solution: Use L'Hospital's Rule (0/0) twice: $\lim_{x\to 0} \frac{\sin(x) - x}{e^x - x - 1} = \lim_{x\to 0} \frac{\cos(x) - 1}{e^x - 1} = \lim_{x\to 0} \frac{-\sin(x)}{e^x} = 0$

d. Suppose $f(x) = \begin{cases} 3x & \text{when } x \le -1 \\ x-3 & \text{when } -1 < x \le 2 \\ x^2-3 & \text{when } 2 < x \end{cases}$

i) $\lim_{x\to -1^-} f(x) =$ __-3__ ii) $\lim_{x\to 2^+} f(x) =$ __1__

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11. (10 points) Suppose a hollow spherical rubber ball has **a radius of 10 centimeters** and is only **.15 cm** thick. **Using the differential [or the linear estimating function], estimate the volume of rubber used to make the ball. You may leave your answer in terms of $\pi$.**

[The volume of a sphere of radius $r$ is $\frac{4}{3}\pi r^3$ .]

Solution: The volume of rubber used to make the ball is estimated by dV when r = 10 and dr = .15.

$V=\frac{4}{3}\pi r^3$ ; $dV=4\pi r^2\, dr$ So dV = 4π 100 (.15) = 60 π

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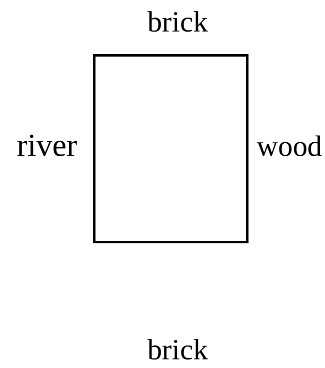
12. (10 points) A rectangular garden is to be constructed bounded on one side by a river bank, on two sides by a brick wall costing **$20.00 per meter** and on the fourth side opposite the river by a wooden fence costing **$10.00 per meter. There is a budget of $400 for the project.**

a.**Find the area of the garden if the total budget is spent and the wooden side is 5 meters long.**




Solution: The wood costs 5*10 = $50, leaving $350 for the 2 brick sides at $20 per meter this gives 17.5 meters for the 2 sides so one side is 8.75 meter. The area is 5 * 8.75 = 43.75 square meters.

a. Ans. The area is 43.75 square meters.

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b.**Use calculus to find the dimensions** of the garden that will **<u>maximize the area</u>** of the garden **<u>subject to the budget constraint of spending $400.</u>**

**Solution: A = b*w , 10w + 20*2*b = 400 so b = 1/40 * (400 - 10 w) ; so A = 1/40( 400w -10w*w)**

**dA/dw = 1/40 (400- 20w) = 0 when w = 20.**

The dimensions of the garden walls that maximizes the enclosed area are **20 m for the wood and 5 meters = brick.**

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13. (10 points) A spider is moving along the curve **$Y = 4X^2 - X$** in the first quadrant of the plane so the spider's <u>Y-coordinate is decreasing at a rate of -5 cm/sec at the point (*x*,y)</u> .

a.Based on this statement **write an equation expressing the derivative of the spider's y-coordinate with respect to time, d*y/d*t , in terms of its *x* coordinate and the *x* coordinate derivative with respect to time, *dx/dt*.**

Solution: **d*y/d*t** = (8x -1)*dx/dt. [The chain rule.]

b. How fast (**with respect to time)** is the spider's X - coordinate changing at the point (1,3)?

Solution: When x = 1 and dy/dt = -5, dx/dt = -5/7 cm / sec.

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14. (20 points) Let **$f(x) = 3x^3 - 9x + 2$.**

a.Find the **critical point(s)** of *f*.

solution: **$f'(x) = 9x^2 - 9$**

0 = 9 (x+1)(x-1) so x = -1 or +1.

<u>Ans. x = 1 , x =-1</u>

b.Find the interval(s) for which *f* is an **increasing** function.

***Solution:***

**$f'(x) = 9x^2 - 9$;** so **$f'(0) = -9$; $f'(10) = 900 - 9$; $f'(-10) = 900 - 9$** so *f* is increasing for (-∞,-1) and (1,∞).

c. Find the interval(s) for which the graph of *f* is **concave up.**

**Solution:** *f* ''(x) = 18 *x* so *f is concave up for the interval (0,∞).*

15. ( 10 points).
A space ship returning to earth from the planet YXO has an **acceleration at time t hours of a(t)=100 km/hr$^2$**. **After three hours,** the **velocity of the ship is 600 km/hour**.

a.Find the **initial velocity** of the ship.
Solution: v(t) = 100 t +C. v(3) = 300 + C = 600 so C = 300.
v(0) = C = 300.

Ans. The **initial velocity** of the ship is 300 km/hour

b. Find the **distance** the ship traveled **during the first hour.**
$S(t) = 50 t^2 + 300 t$; S(1) = 350.

Ans. 350 km.

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16. (10 points) Suppose $f(x) = e^{x/2}$.
a.Find the linear estimating polynomial L($x$) and the quadratic estimating polynomial Q($x$) for $f$ **centered at $x$ = 0.**

$f'(x) = 1/2e^{x/2}$ ; $f''(x) = 1/4e^{x/2}$
$f'(0) = 1/2$ ; $f''(0) = 1/4$

a) Ans. L($x$) = 1+ ½ $x$ ; Q($x$) = 1+ ½ $x$ + 1/4 *1/2 $x^2$

b.Use Q($x$) the quadratic polynomial found in part a) to estimate $f(1) = \sqrt{e}$ .

*Solution:* $f(1) = \sqrt{e} \approx$ Q(1) = 1+ ½ + 1/4 *1/2 = 1 5/8 = 1.625
b) Ans. f(1) ≈ 1.625

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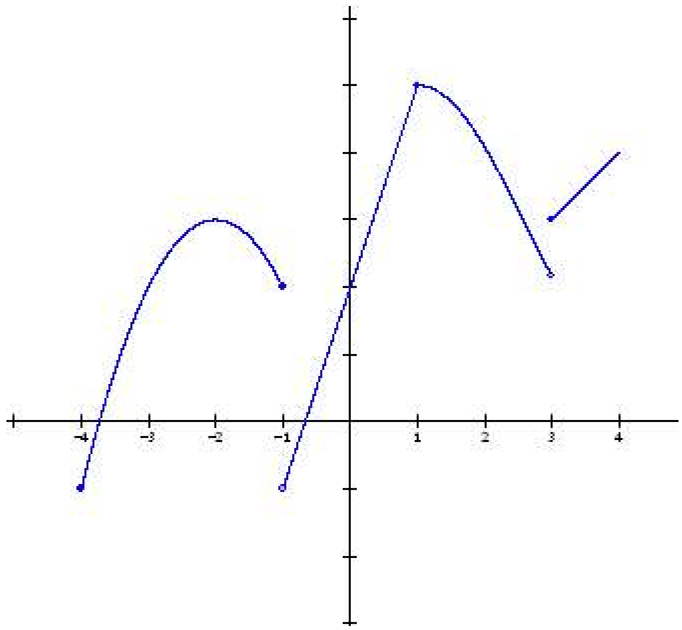
17. (10 points) True or False. Circle the most appropriate response
For this problem assume that P($x$) is the function with the following graph:




Graph of P($x$)

a. P is continuous at $x$ = 3.....................F.

b. P is continuous at $x$ = 1...................T.

c. P is differentiable at $x$ = 2................T.

d. P is differentiable at $x$ = 1 .............F.

e. If 1.5 < $x$ < 2.5 then P'($x$) ≥ 0 ..... F.

f. P "(-2) ≥ 0 ....................................F.