Using Mapping Diagrams to Understand Trigonometric Functions
AMATYC Webinar
April 10, 2014
Martin Flashman

http://users.humboldt.edu/flashman/AMATYC/AMATYC_TRIG_LINKS.html

Professor of Mathematics
Humboldt State University
flashman@humboldt.edu
http://users.humboldt.edu/flashman

Background Questions

Hands Up or Down…

1. Are you familiar with Mapping Diagrams?
2. Have you watched “Using Mapping Diagrams to Understand Functions”? from AMATYC
3. Have you used Mapping Diagrams in your teaching?

Outline

I. Mapping Diagram Basics
II. Triangle Trigonometry
III. Circle Trigonometry
IV. Periodicity and Symmetry Identities
V. Solving Equations: Inverse Trigonometry
VI. Composition

I. Mapping Diagram Basics

A.k.a.

Function Diagrams

Dynagraphs

Preface: Trigonometric Example Will be reviewed at end. ☺

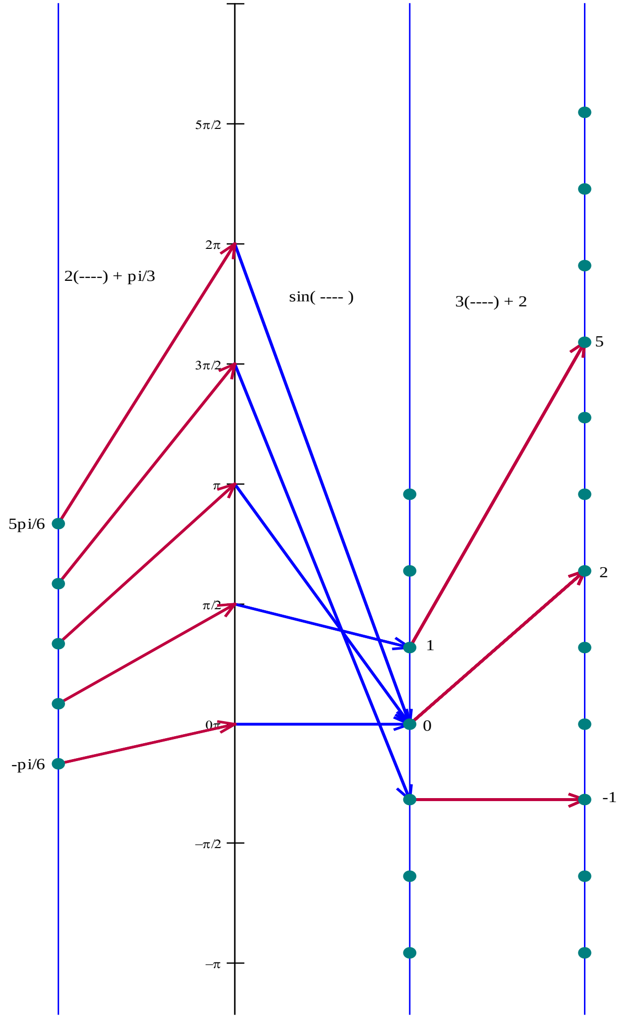
$y = g(x) = 3 \sin(2x + \pi/3) + 2$

Consider steps for *g* <u>*(as a composition)*</u> :

1. **Linear:** **multiply x by 2 then add π/3:** $u = 2x + \pi/3$.
2. **Find sine of result:** $v = \sin(u)$
3. **Linear: Multiply *v* by 3 then add 2 so**

   $y = 3v + 2$

Mapping diagram



$y = g(x) = 3\sin(2x + \pi/3) + 2$

Before: $u = 2x + \pi/3$.
MIDDLE: $v = \sin(u)$
After: $y = 3v + 2$

Main Resource for Future

- Mapping Diagrams from A (algebra) B(asics) to C(alculus) and D(ifferential) E(quation)s. A Reference and Resource Book on Function Visualizations Using Mapping Diagrams (Preliminary Sections- NOT YET FOR publication)

- http://users.humboldt.edu/flashman/MD/section-1.1VF.html

Linear Mapping diagrams

We begin our more detailed introduction to mapping diagrams by a quick consideration of linear functions :

" $y = f(x) = mx + b$ "

You can download and try the worksheet : Worksheet.VF1.pdf.

Linear Functions: Tables

x	2x + 3
3	
2	
1	
0	
-1	
-2	
-3	

Complete the table.

x = 3,2,1,0,-1,-2,-3

f(x) = 2x + 3

f(0) = ____?

For which x is f(x) > 0?

Linear Functions: Tables

x	2x + 3
3	9
2	7
1	5
0	3
-1	1
-2	-1
-3	-3

Complete the table.

$x = 3,2,1,0,-1,-2,-3$

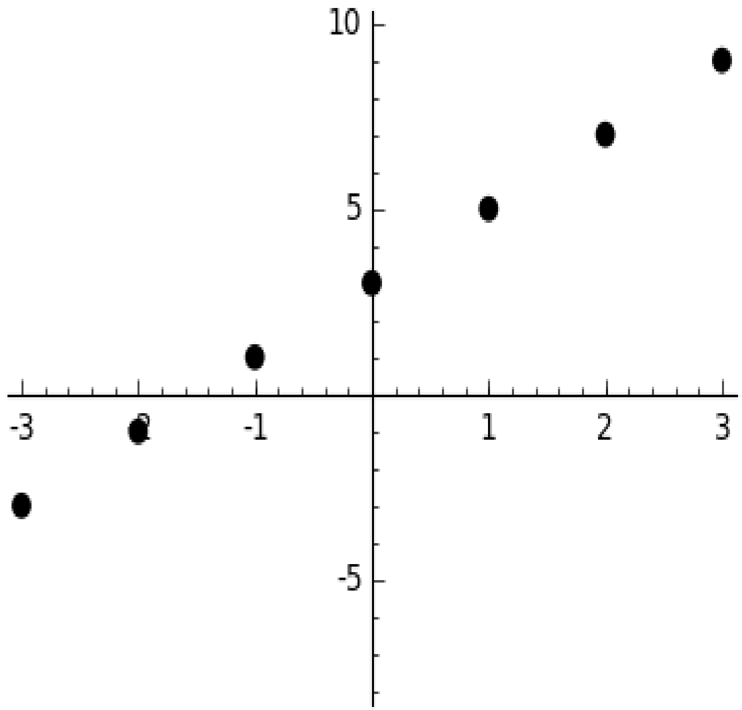
$f(x) = 2x + 3$

$f(0) =$ ____?

For which x is $f(x) > 0$?

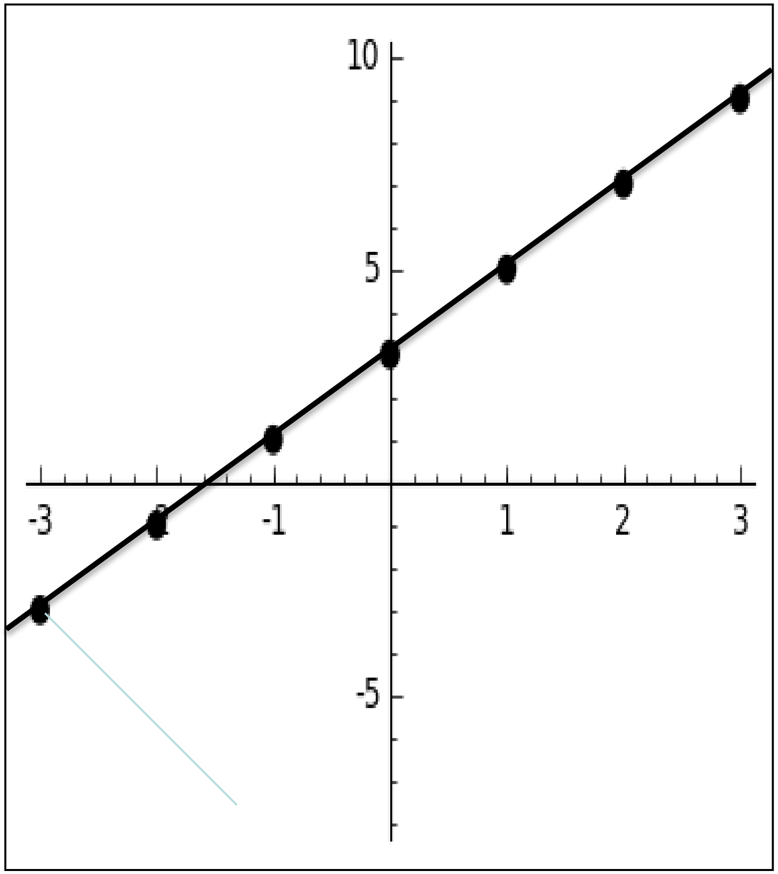
Linear Functions: On Graph

Plot Points (x , 2x + 3):



x	2x + 3
3	9
2	7
1	5
0	3
-1	1
-2	-1
-3	-3

Linear Functions: On Graph



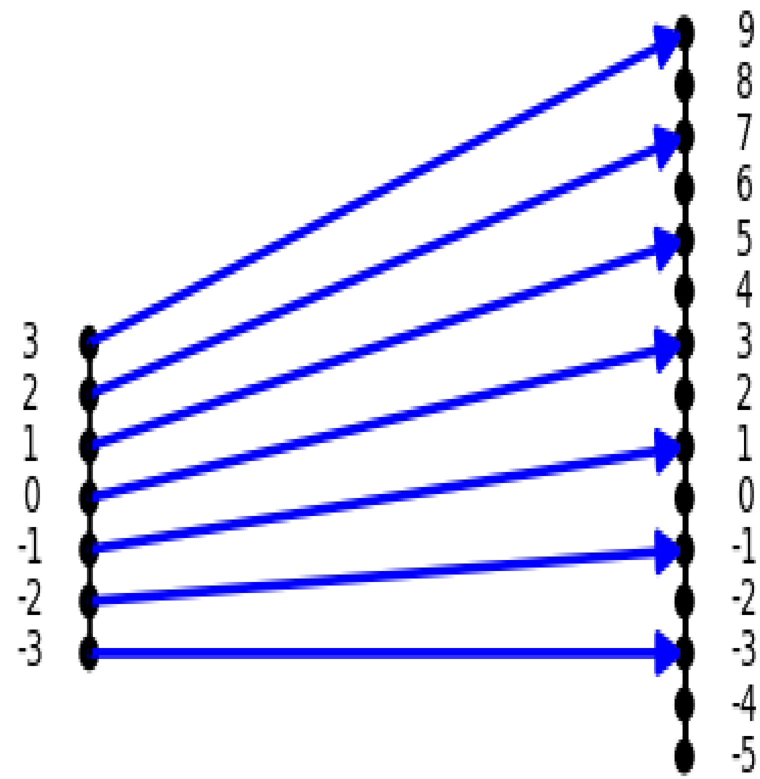
Connect Points
$(x , 2x + 3)$:

x	2x + 3
3	9
2	7
1	5
0	3
-1	1
-2	-1
-3	-3

Linear Functions: Mapping diagrams What happens before the graph.

- Connect point x to point 2x + 3 on axes

x	2x + 3
3	9
2	7
1	5
0	3
-1	1
-2	-1
-3	-3



Examples on Excel / Geogebra / SAGE

- Excel example
- Geogebra example

You can download the trig worksheet: **TrigWorksheet.AMATYC.pdf**

Why Trig- How Trig

- (Right)Triangles
- Location
  - Surveying
  - Navigation
  - Astronomy
- Functions and Modeling
  - Sound
  - Light

- Key Right Triangles
  - 30°, 45°, 60°
  - 3,4,5; 5,12,13
- Unit Circle
- Tables
- Slide Rules
- e-Calculators
- Graphs
- Mapping Diagrams!

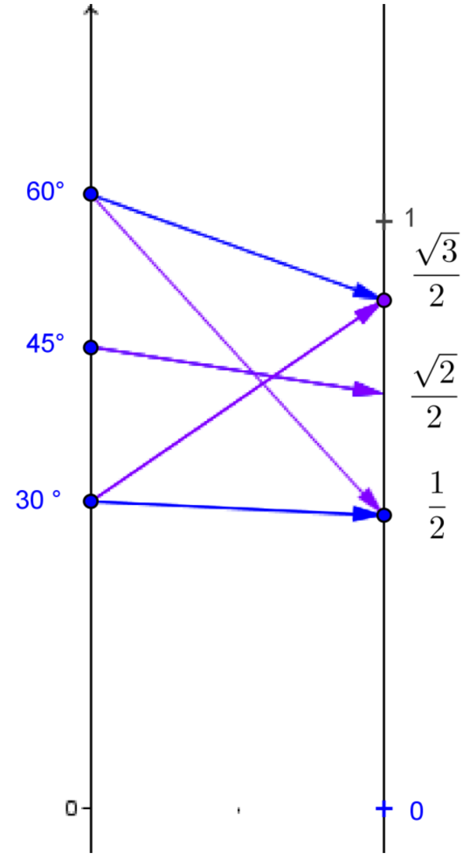
II. Triangle Trigonometry

Right Triangle Trigonometry
[GeoGebra]

First Table of Values

Θ: Angle in Degrees	sin(θ)	cos(θ)
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

Simple Mapping Diagram



Right Triangle Table

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\text{cosec}\ \theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

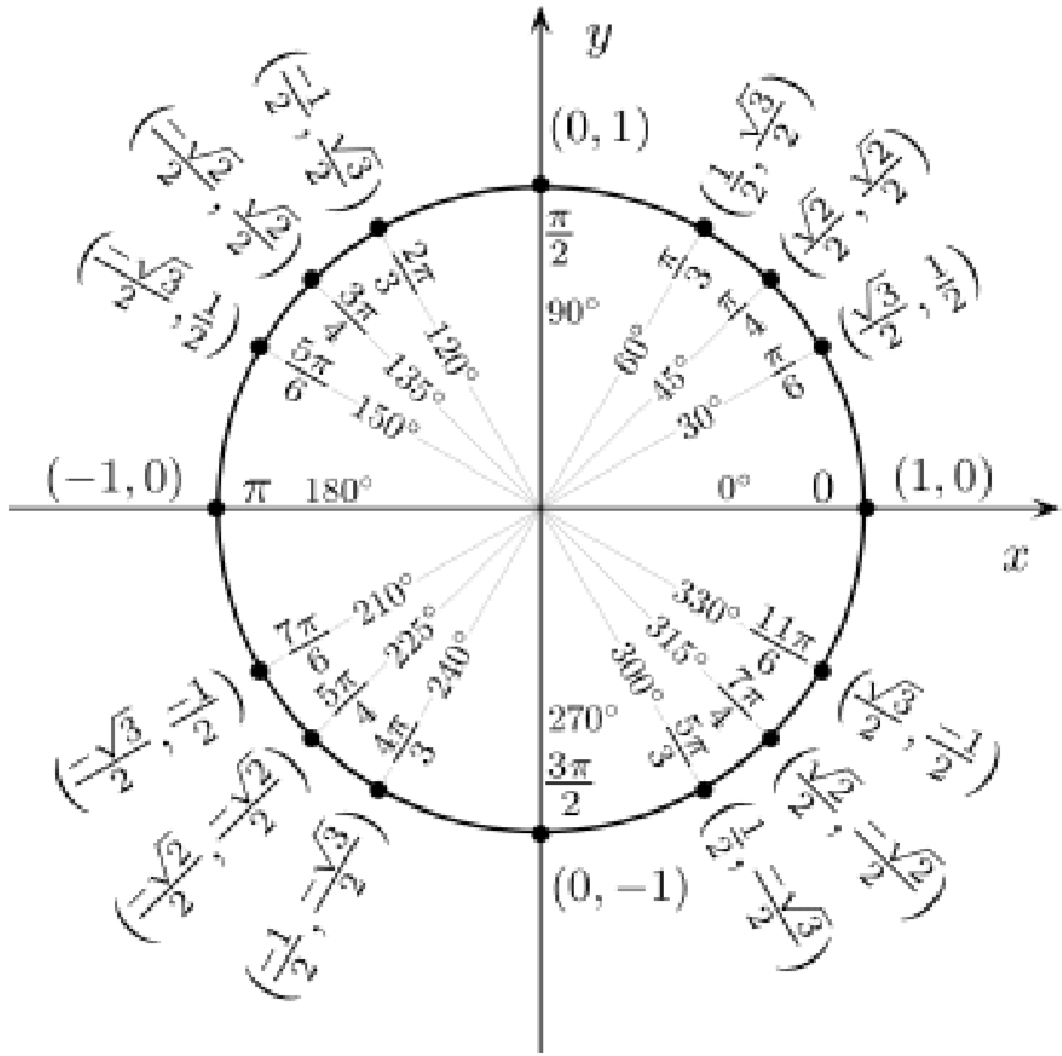
Natural Trigonometric Functions

θ		Sin	Cos	Tan
Deg	Rad			
0	0.000	0	1	0
1	0.017	0.0175	0.9998	0.0175
2	0.035	0.0349	0.9994	0.0349
3	0.052	0.0523	0.9986	0.0524
4	0.070	0.0698	0.9976	0.0699
5	0.087	0.0872	0.9962	0.0875
6	0.105	0.1045	0.9945	0.1051
7	0.122	0.1219	0.9925	0.1228
8	0.140	0.1392	0.9903	0.1405
9	0.157	0.1564	0.9877	0.1584
10	0.174	0.1736	0.9848	0.1763
11	0.192	0.1908	0.9816	0.1944
12	0.209	0.2079	0.9781	0.2126
13	0.227	0.2250	0.9744	0.2309
14	0.244	0.2419	0.9703	0.2493
15	0.262	0.2588	0.9659	0.2679
16	0.279	0.2756	0.9613	0.2867
17	0.297	0.2924	0.9563	0.3057
18	0.314	0.3090	0.9511	0.3249
19	0.331	0.3256	0.9455	0.3443
20	0.349	0.3420	0.9397	0.3640
21	0.366	0.3584	0.9336	0.3839
22	0.384	0.3746	0.9272	0.4040
23	0.401	0.3907	0.9205	0.4245
24	0.419	0.4067	0.9135	0.4452
25	0.436	0.4226	0.9063	0.4663
26	0.454	0.4384	0.8988	0.4877
27	0.471	0.4540	0.8910	0.5095
28	0.488	0.4695	0.8829	0.5317
29	0.506	0.4848	0.8746	0.5543
30	0.523	0.5000	0.8660	0.5774
31	0.541	0.5150	0.8572	0.6009
32	0.558	0.5299	0.8480	0.6249
33	0.576	0.5446	0.8387	0.6494
34	0.593	0.5592	0.8290	0.6745
35	0.611	0.5736	0.8192	0.7002
36	0.628	0.5878	0.8090	0.7265
37	0.645	0.6018	0.7986	0.7536
38	0.663	0.6157	0.7880	0.7813
39	0.680	0.6293	0.7771	0.8098
40	0.698	0.6428	0.7660	0.8391
41	0.715	0.6561	0.7547	0.8693
42	0.733	0.6691	0.7431	0.9004
43	0.750	0.6820	0.7314	0.9325
44	0.768	0.6947	0.7193	0.9657
45	0.785	0.7071	0.7071	1.0000

θ		Sin	Cos	Tan
Deg	Rad			
46	0.802	0.7193	0.6947	1.0355
47	0.820	0.7314	0.6820	1.0724
48	0.837	0.7431	0.6691	1.1106
49	0.855	0.7547	0.6561	1.1504
50	0.872	0.7660	0.6428	1.1918
51	0.890	0.7771	0.6293	1.2349
52	0.907	0.7880	0.6157	1.2799
53	0.925	0.7986	0.6018	1.3270
54	0.942	0.8090	0.5878	1.3764
55	0.959	0.8192	0.5736	1.4281
56	0.977	0.8290	0.5592	1.4826
57	0.994	0.8387	0.5446	1.5399
58	1.012	0.8480	0.5299	1.6003
59	1.029	0.8572	0.5150	1.6643
60	1.047	0.8660	0.5000	1.7321
61	1.064	0.8746	0.4848	1.8040
62	1.082	0.8829	0.4695	1.8807
63	1.099	0.8910	0.4540	1.9626
64	1.116	0.8988	0.4384	2.0503
65	1.134	0.9063	0.4226	2.1445
66	1.151	0.9135	0.4067	2.2460
67	1.169	0.9205	0.3907	2.3559
68	1.186	0.9272	0.3746	2.4751
69	1.204	0.9336	0.3584	2.6051
70	1.221	0.9397	0.3420	2.7475
71	1.239	0.9455	0.3256	2.9042
72	1.256	0.9511	0.3090	3.0777
73	1.273	0.9563	0.2924	3.2709
74	1.291	0.9613	0.2756	3.4874
75	1.308	0.9659	0.2588	3.7321
76	1.326	0.9703	0.2419	4.0108
77	1.343	0.9744	0.2250	4.3315
78	1.361	0.9781	0.2079	4.7046
79	1.378	0.9816	0.1908	5.1446
80	1.396	0.9848	0.1736	5.6713
81	1.413	0.9877	0.1564	6.3138
82	1.430	0.9903	0.1392	7.1154
83	1.448	0.9925	0.1219	8.1443
84	1.465	0.9945	0.1045	9.5144
85	1.483	0.9962	0.0872	11.4301
86	1.500	0.9976	0.0698	14.3007
87	1.518	0.9986	0.0523	19.0811
88	1.535	0.9994	0.0349	28.6363
89	1.553	0.9998	0.0175	57.2900
90	1.570	1.0000	0.0000	

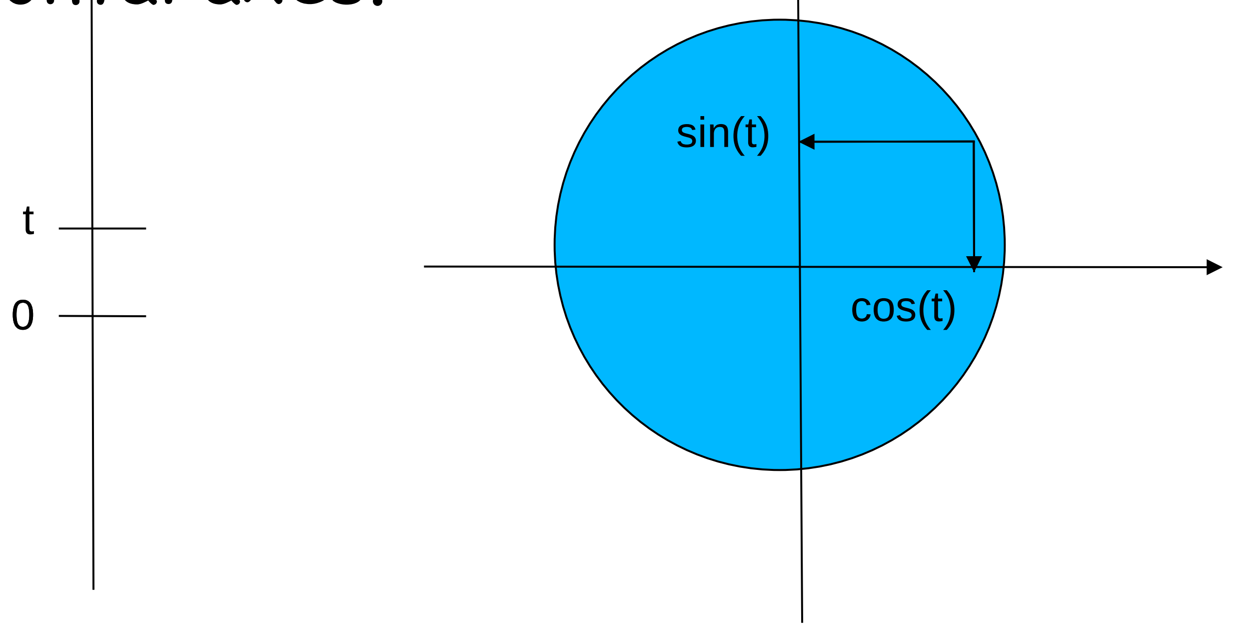
III Circle Trigonometry

Unit Circle Chart (Wikipedia)



Seeing the functions on the unit circle with mapping diagram.

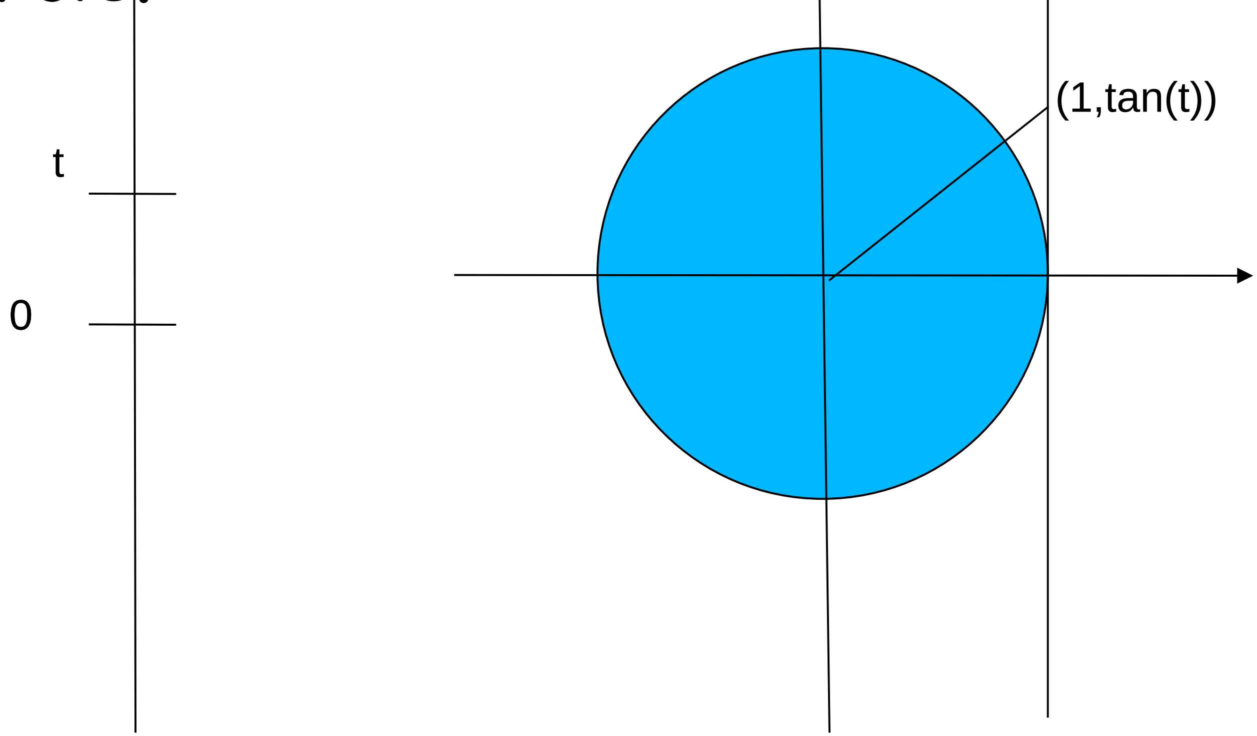
Sine and cosine of t measured on the vertical and horizontal axes.



Note the visualization of periodicity.

Tangent Interpreted on Unit Circle

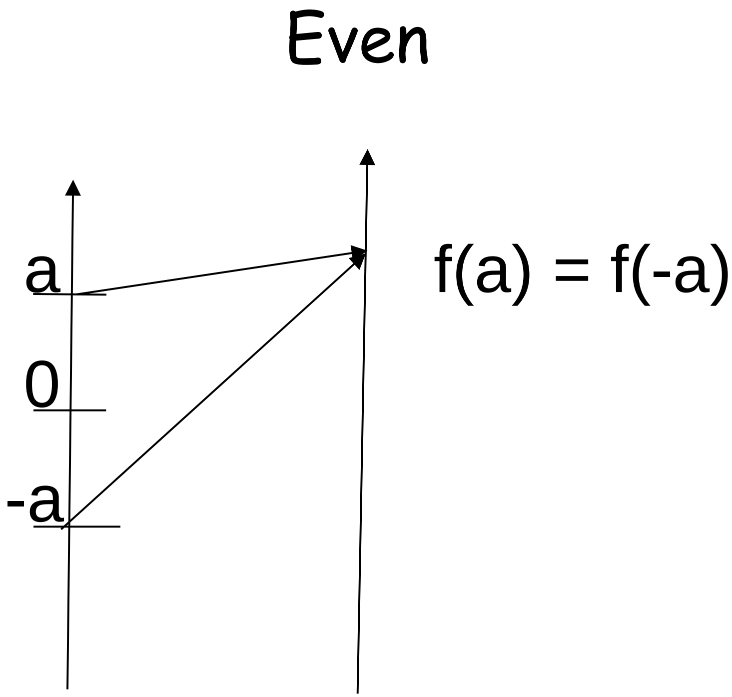
- tan(t) measured on the axis tangent to the unit circle.

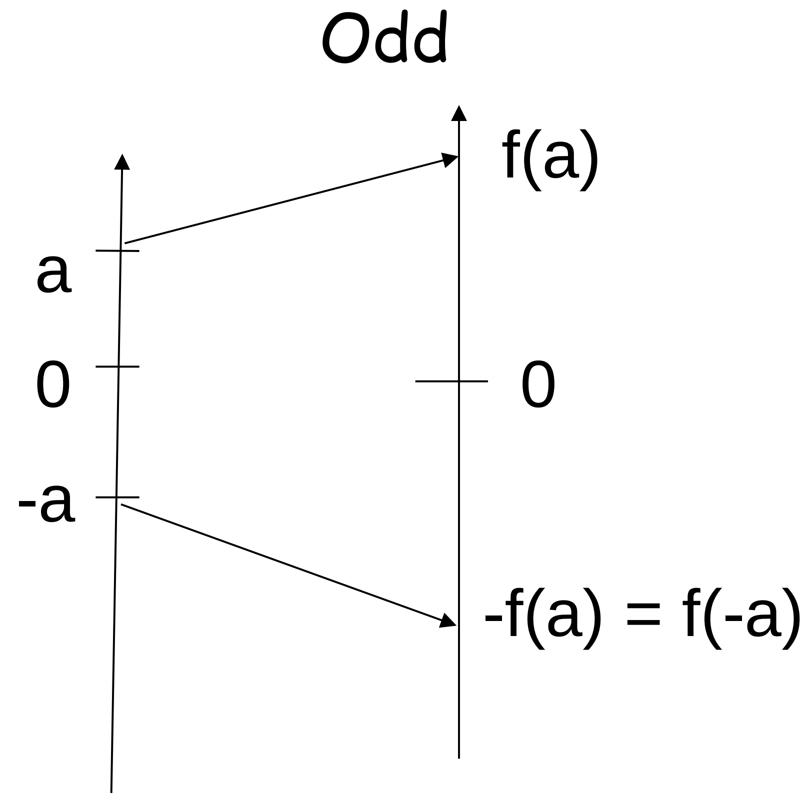


- Note the visualization of periodicity.

IV Periodicity and Symmetry Identities

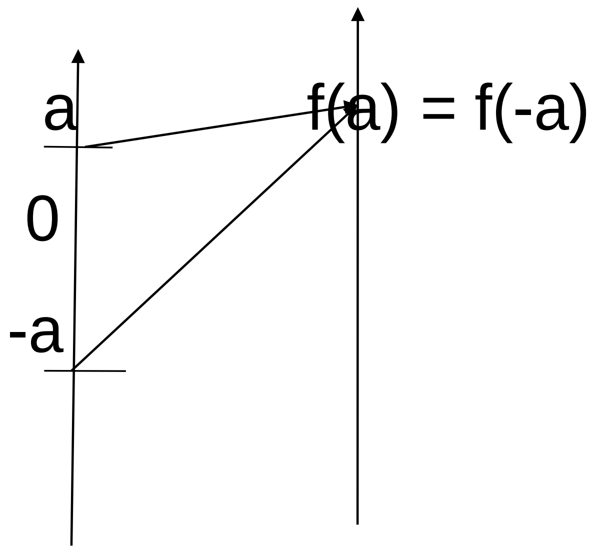
Even and odd on Mapping diagrams

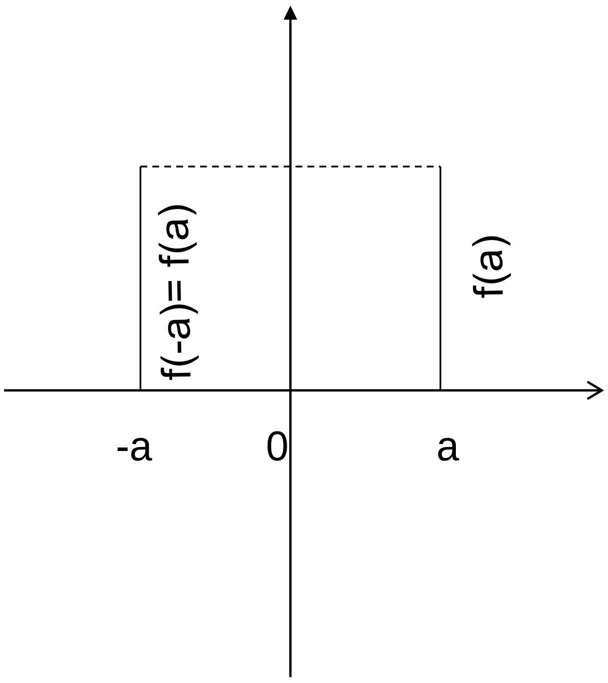




Even Function Mapping Figures and Graphs

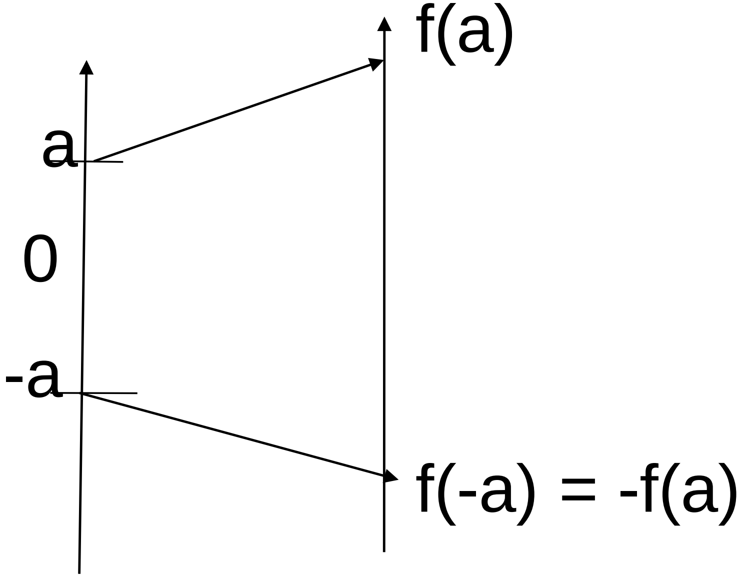
An Even Function

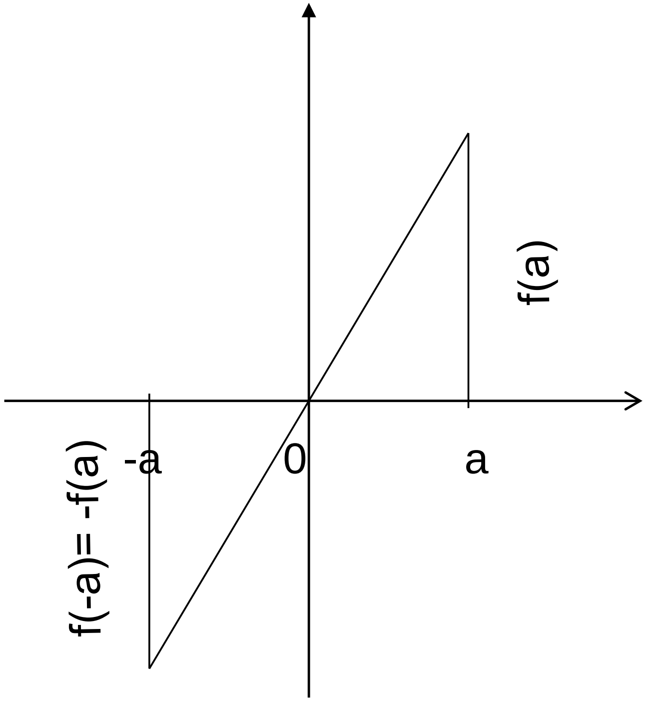




Odd Function Mapping Figures and Graphs

An Odd Function



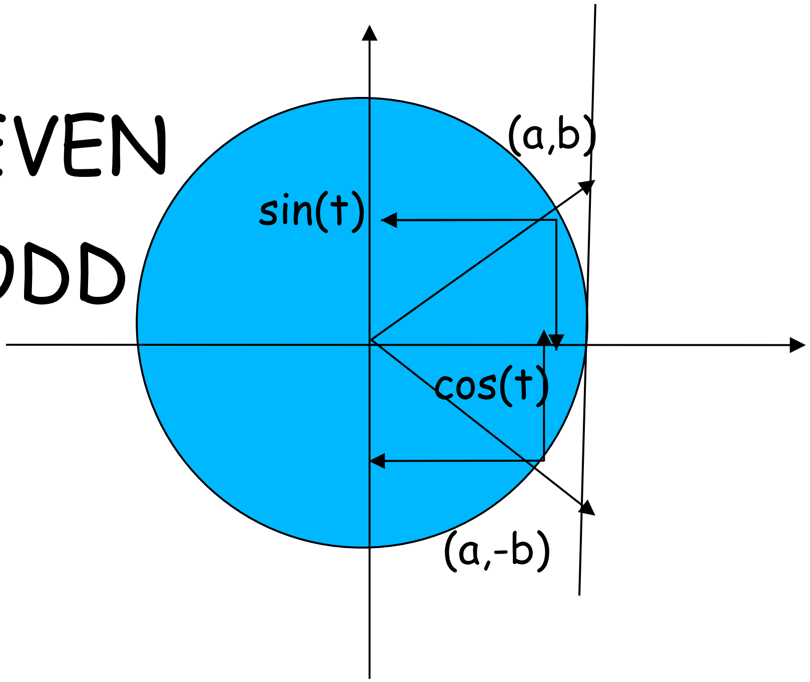


Trigonometric functions and symmetry:

cos(-t) = cos(t) for all t. EVEN

sin(-t) = -sin(t) for all t. ODD

tan(-t) = - tan(t) for all t.



Justifications from unit circle mapping diagrams for sine, cosine and tangent..

V. Solving Equations: Inverse Trigonometry

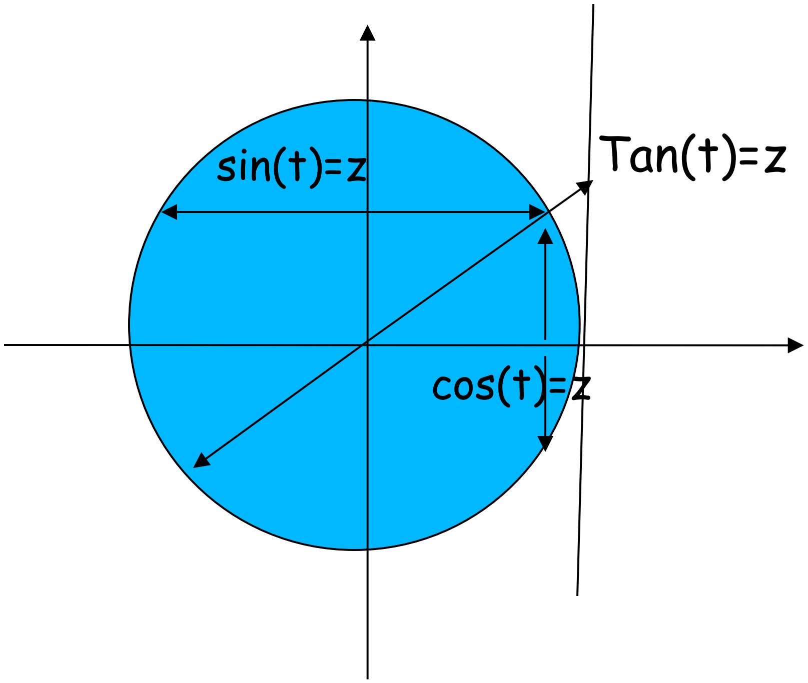
Trig Equations and Mapping diagrams

- To solve ***trig(x)* = z**.
- Find z on the target axis, then trace back on any and all arrows that "hit" z.
- Notice how this connects to periodic behavior of the trig functions and the issue of the number of solutions in an interval.
- This also connects to understanding the inverse trig functions.

Solving Simple Trig Equations:

Solve trig(t)=z from unit circle mapping diagrams for sine, cosine and tangent.

[View on GeoGebra]



Inverses, Equations and Mapping diagrams

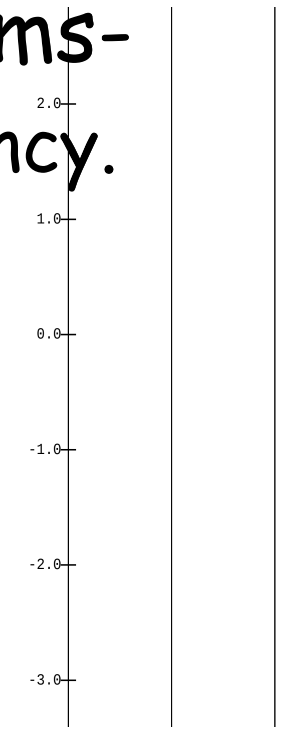
- Inverse in general:
  - If f(x) = y then invf(y)=x.
- So to find invf(b) we need to find any and all x that solve the equation f(x) = b.
- How is this visualized on a mapping diagram?
- Find b on the target axis, then trace back on any and all arrows that "hit"b.

Mapping diagrams and Inverses

Inverse linear functions:

- Use transparency for mapping diagrams-
  - Copy mapping diagram of f to transparency.
  - Flip the transparency to see mapping diagram of inverse function g. ("before or after")

    invg(g(a)) = a;   g(invg(b)) = b;
- Example i: g(x) = 2x; invg(x) = 1/2 x
- Example ii: h(x) = x + 1 ; invh(x) = x - 1



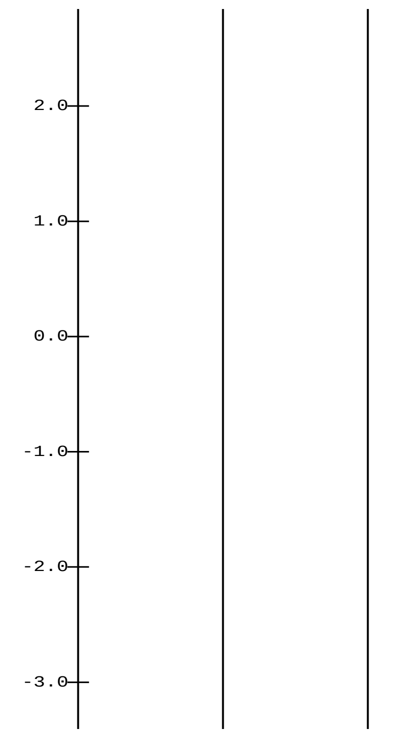
Mapping diagrams and Inverses

Socks and shoes with mapping diagrams

- $g(x) = 2x$; $invg(x) = 1/2\ x$
- $h(x) = x + 1$; $invh(x) = x - 1$

- $f(x) = 2\ x + 1 = (2x) + 1$
  - $g(x) = 2x$; $h(u)=u+1$
  - inverse of f:

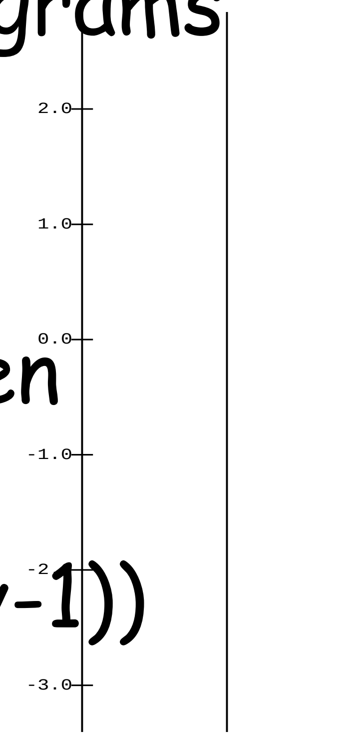
  $invf(x)=invh(invg(x))=1/2(x-1)$



Mapping diagrams and Inverses

Inverse trigonometric functions:

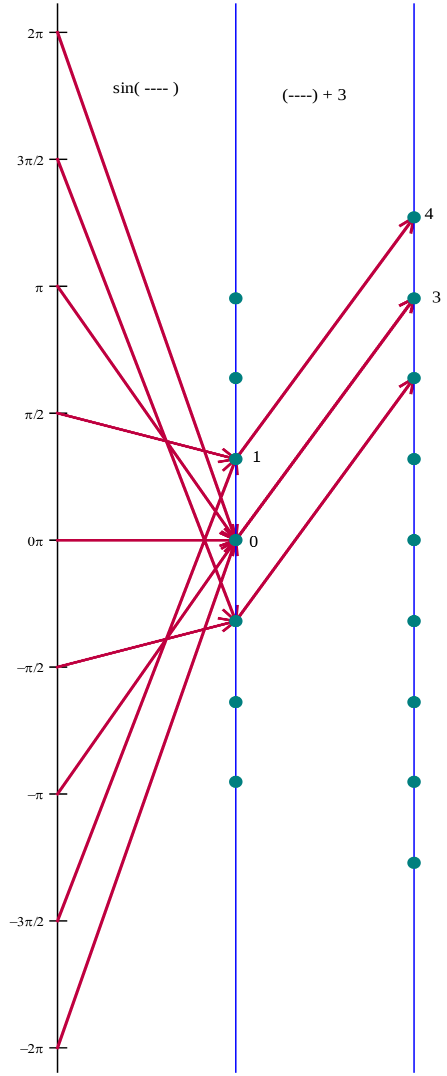
- socks and shoes with mapping diagrams
- $g(x) = \tan(x)$; $invg(x) = \arctan(y)$
- $h(u) = 2u + 1$; $invh(u) = (u-1)/2$
- If $f(x) = 2 \tan(x) + 1 = h(g(x))$ then
- inverse of f: $invf(x) = invh(invg(y)) = \arctan(1/2(y-1))$



VI. Compositions with Trig Functions

[GeoGebra]

Example: y = f(x) =sin(x) + 3



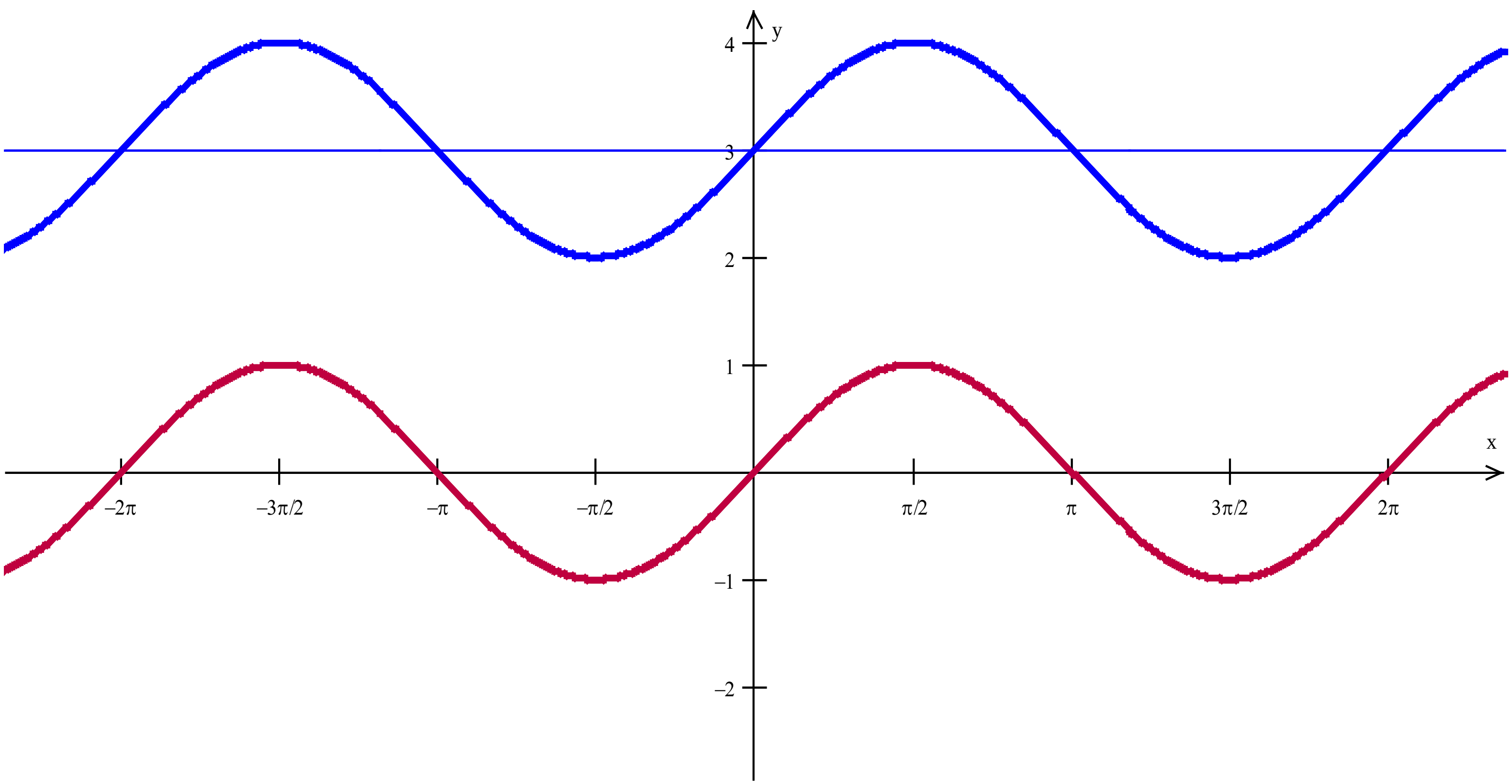
- Mapping diagram for
- y =f(x) = sin(x) + 3 considered as a composition:
- First: u = sin(x)
- Second: y =u+3 so the result is
- y = (sin(x)) + 3

Example: Graph of y = sin(x) + 3

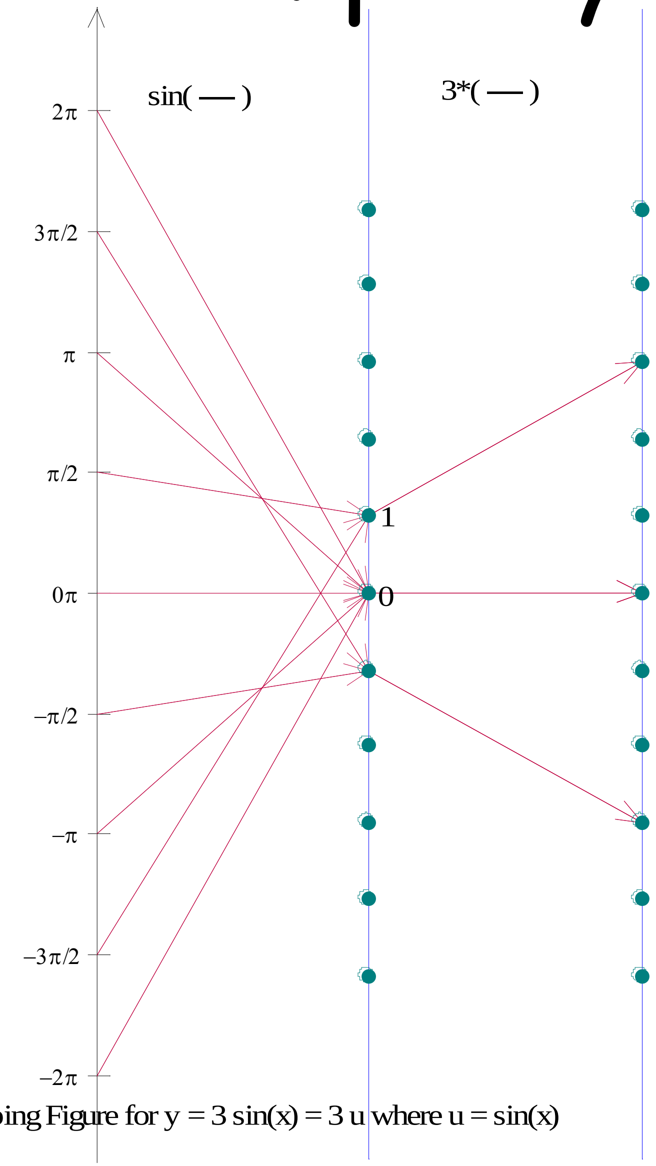
Graph of y = sin(x) + 3 [Winplot]

Amplitude: 1

Period: 2π



Example: y = f(x) = 3 sin(x)



Mapping Figure for y = 3 sin(x) = 3 u where u = sin(x)

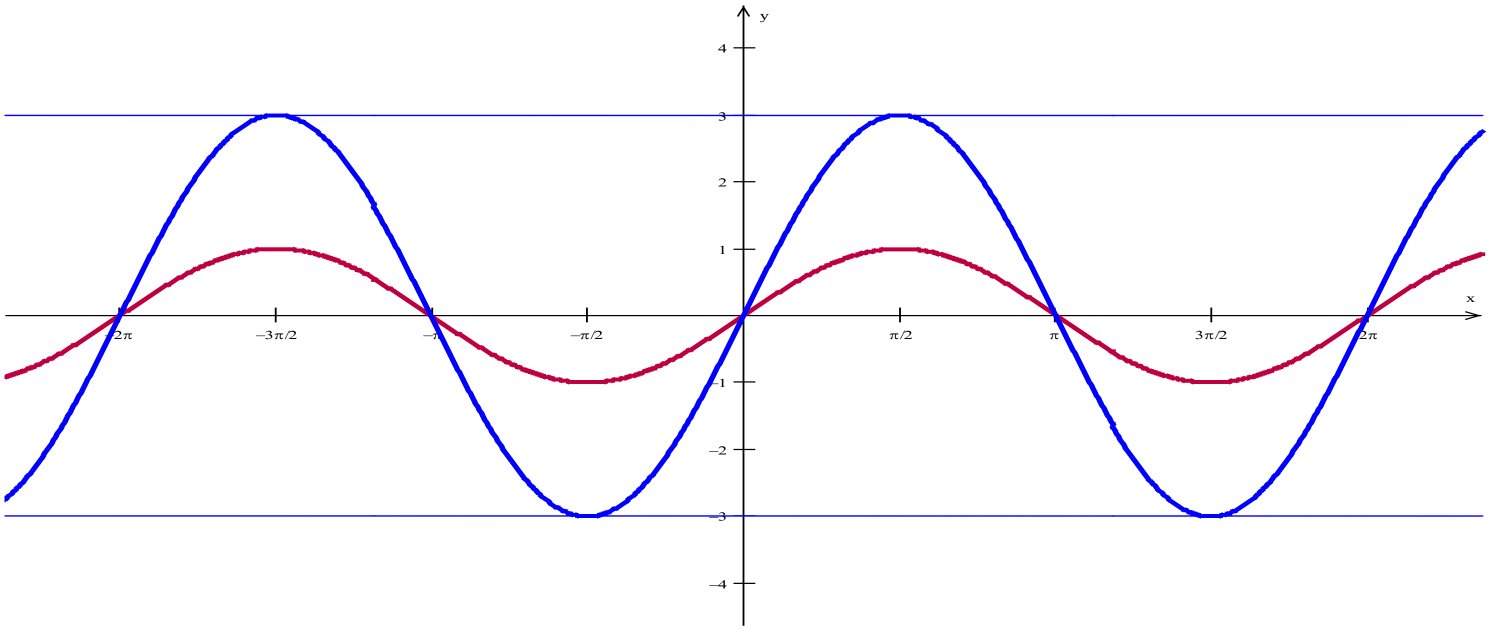
- Mapping diagram for
- y =f(x) = 3 sin(x) considered as a composition:
- First: u = sin(x)
- Second: y = 3u so the result is
- y = 3 (sin(x))

Example: Graph of $y = 3 \sin(x)$

Graph of $y = 3 \sin(x)$ [Winplot]

Amplitude: 3

Period: 2π



Scale change before trig.

Mapping figures and graphs for $f(x) = \sin(Bx)$

- Amplitude and period

Connection to solving equations:

- Example: $\sin(2x) = 1$;
  - $2x = \pi/2, 5\pi/2$
  - $x = \pi/4 , 5\pi/4$
  - Difference is period: $(5\pi - \pi)/4 = \pi$.

Scale change before trig

Mapping figures and graphs for $f(x) = \sin(x + D)$ or

- Amplitude and period and shift.

Connection to solving equations:

- Example: $\sin(x + \pi/3) = 0$;
  - $x + \pi/3 = 0$
  - $x = -\pi/3$

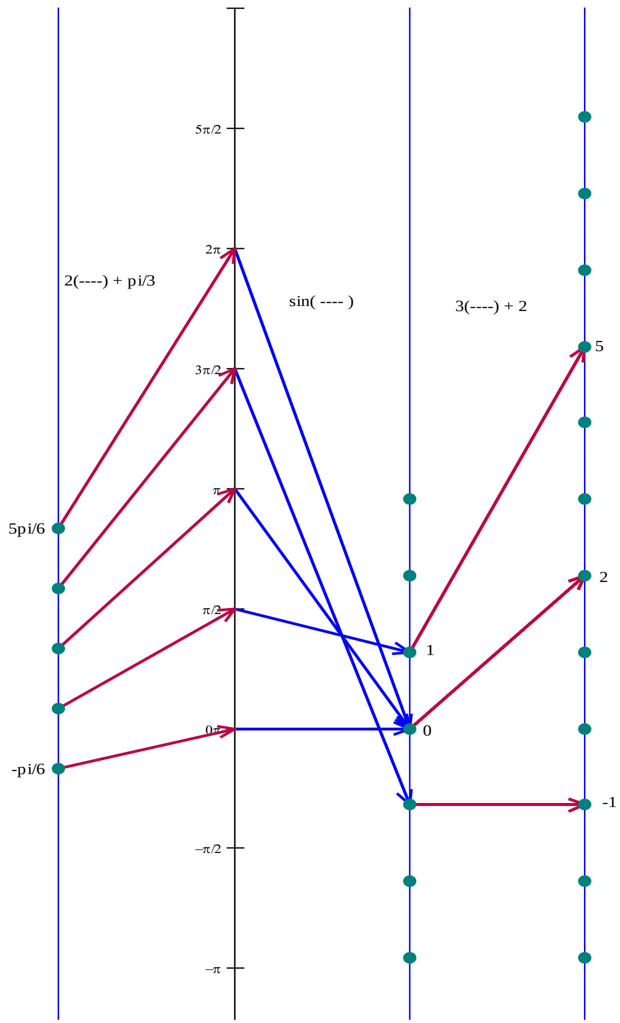
Shift of sine curve to start at $x = -\pi/3$: $(-\pi/3, 0)$

- Interpretations of these functions with circles.

Altogether!

- $f(x) = 3 \sin(2x+\pi/3) + 2$
- Mapping figure: Before $u = 2x+\pi/3$
- After $y = 3z + 2$
- MIDDLE: $z = \sin(u)$.
- Amplitude :3, period: π , and shift: ???.
- Visualize on circle. Dot races and mapping figures.
- Solve equations for period and shift.
- $u = 0$ and $u = 2\pi$. Period = difference in x.

Mapping diagram



$f(x) = 3\sin(2x+\pi/3) + 2$

Mapping figure:

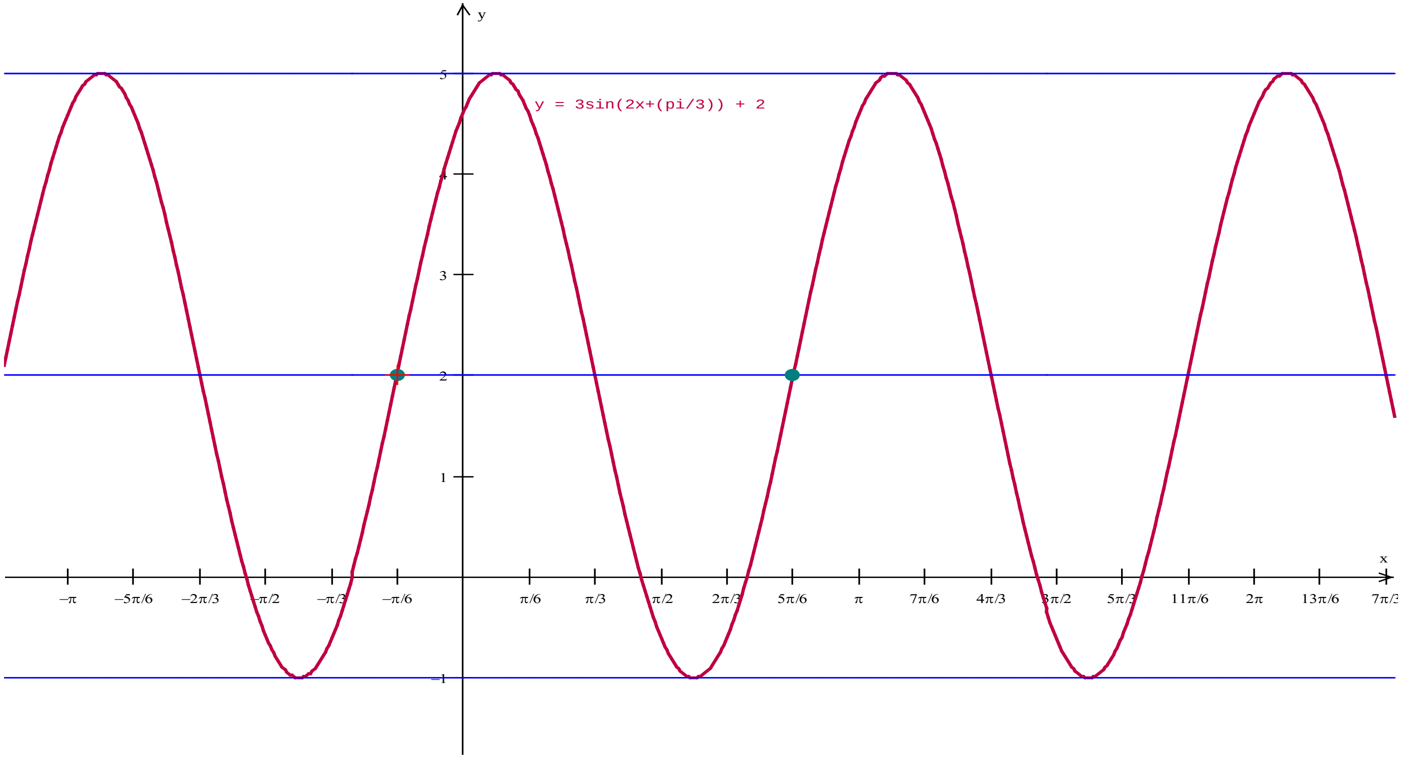
Before $u = 2x+\pi/3$

After $y = 3z + 2$

MIDDLE: $z = \sin(u)$.

Graph

- $f(x) = 3\sin(2x+\pi/3) + 2$



Thanks
The End!

☺

Questions?

flashman@humboldt.edu
http://www.humboldt.edu/~mef2

References

http://users.humboldt.edu/flashman/AMATYC/AMATYC_TRIG_LINKS.html

Main Resource for Future

- **Flashman**

Mapping Diagrams from A (algebra) to C(alculus) and D(ifferential) E(quation)s. A Reference and Resource Book on Function Visualizations Using Mapping Diagrams

(Preliminary Sections- NOT YET FOR publication)

http://users.humboldt.edu/flashman/MD/section-1.1VF.html

Mapping Diagrams and Functions

- **Flashman,** AMATYC Webinar, "Using Mapping Diagrams to Understand Functions" (Watch on youtube) October 15, 2013.
- **Flashman,** "Concepts to Drive Technology in the 21st Century" (html), **ComputerBasedMath** ™(YouTube Video) (CBM) Education Summit, UNICEF, NYC, Nov.22, 2013.
- **Flashman,** **"The Sensible Calculus Program". (Prospectus)**
- **Henri Picciotto**, Function Diagrams
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- **Flashman, Yanosko, Kim** https://www.math.duke.edu//education/prep02/teams/prep-12/

More References

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Thanks

The End! REALLY!

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flashman@humboldt.edu

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