I.I.2 Intermediate Values

Motivation: If the temperature was 30 degrees centigrade at 1 P.M. and 20 degrees at 5 P.M., it seems reasonable to say that at some time between 1 and 5 in the afternoon, the temperature was exactly 27.4524375 degrees.

On the other hand if the cost of a certain stock on the stock exchange was $30 a share at 1 P.M. and $20 a share at 5 P.M. , it would be absurd to claim that the cost of a single share was ever $27.4524375. These two similar statements with their quite different conclusions illustrate the nature and relevance of asking a question of intermediate values for a function.

The question of intermediate values for a function is itself quite simple to state. **If *f*(*a*) = R and *f*(*b*) = T and S is a value between R and T, is there a number c so that *f*(c) = S?** The question can be extended by asking for a precise value for c or at least a good approximation . It is not always possible to find a solution to this problem as the following example illustrates.

Example I.I.5: Suppose $f(x) = \begin{cases} 3 \text{ if } x \le 2 \\ 5 \text{ if } x > 2 \end{cases}$. Then 4 is between *f*(1) and *f*(3), but there is no number c for which *f*(c) = 4. Note that this function is not continuous at *x* = 2. (See Figure ***.). The example illustrates that when a function is not continuous, the values can have gaps between them. This does not happen when a function is continuous on the interval between *a* and *b*. This fact is expressed by the following

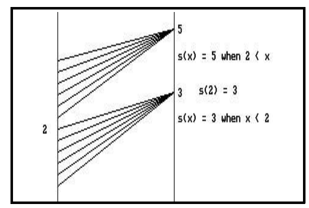

Figure 1

Intermediate Value Theorem: If ***f* is a continuous function on the interval [*a*,*b*] and S is a value between *f*(*a*) and *f*(*b*)** then **there is a number c ∈ [*a*,*b*] where *f*(c) = S.**

Discussion: The proof of this theorem will be presented in detail in an appendix. The visual interpretations in terms of the transformation diagram of a moving object and the graph of the function both add in rather convincing ways to its credibility. For the transformation diagram, think of a light moving in continuous way on the target line so that at time t the light is at point *f*(t). (See Figure ***.) Then at time *a*, the light is at point *f*(*a*), at time *b* the light is at point *f*(*b*) and S is a point between *f*(*a*) and *f*(*b*) on the line. It is the continuity of the light's motion that allows us to say that at some time c between *a* and *b* the light is located at point S, i.e., *f*(c)=S.

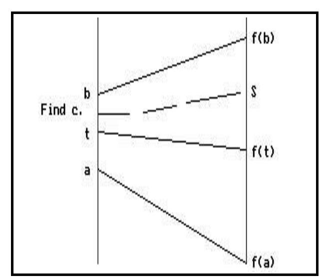

Figure 2

In considering the graph of the function *f*, (see Figure ***), we see that the points (*a*,*f*(*a*)) and (*b*,*f*(*b*)) lie on different sides of the line Y= S because S is between *f*(*a*) and *f*(*b*). For the graph to be continuous it must connect these two points without any breaks in the curve. Thus at some point the graph of *f* must

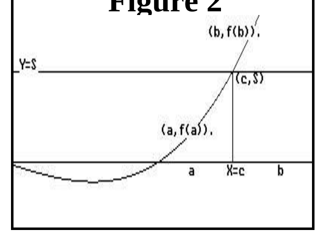

Figure 3

cross the line Y=S. The coordinates of that point are (c,S) to be on the line and also (c,f(c)) to be on the graph of f. So for that value of c, between a and b, f(c)=S.

Application to Solving Equations: Show that the equation $x^3 - 5x + 2 = 0$ has a solution in the interval [0,1].

Solution: Consider the function $f(x) = x^3 - 5x + 2$. This function is continuous on the interval [0,1] and $f(0)=2$ while $f(1) = -2$. Since 0 is between -2 and 2, the Intermediate Value Theorem implies there is a number $c \in [0,1]$ where $f(c) = 0$. That is, c solves the equation $x^3 - 5x + 2 = 0$.

Application to Understanding Inequalities: Suppose g is a continuous function on [-5, 5] and g(x) = 0 **only when x=-3 and x = 4.** If g(-5) = 7, g(0) = -2, and g(7) = 3, then for any $x \in$ [-5,-3) and (4,5] , g(x) > 0. Furthermore, for any $x \in$ (-3, 4), g(x)<0. (See Figure *** below.)

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     g(x):   7         0         -2                 0
  x values: -5   v    -3          0                 4    5
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Figure 4

Discussion: Suppose there is a value v∈[-5,-3) where g(v) < 0. Then since g(v) < 0 < g (-5)=7 and g is continuous on the interval [-5,v], by the Intermediate Value Theorem there should be a point c between -5 and v where g(c) = 0. But we are given that x=-3 and x=4 are the **only** x values for which g(x)=0. Thus it is impossible for g(v) to be less than or equal to 0 for v ∈ [-5,-3). So for v ∈ [-5,-3) we have that g(v)>0 as claimed. Similar arguments using the Intermediate Value Theorem work for the other inequalities .

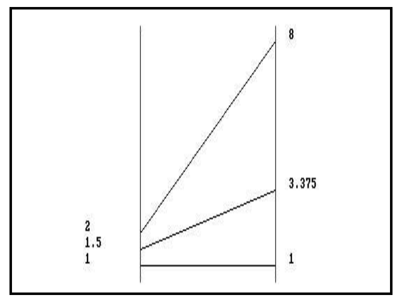


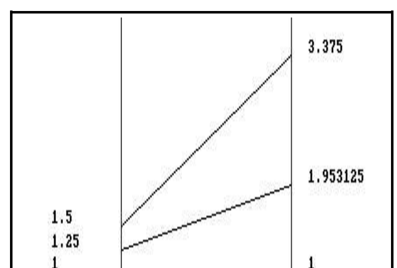
Figure 5

Application: (The Bisection Method) Estimate the solution to the equation $x^3 = 2$ as a decimal number.

Solution: Of course the exact solution to this equation is the cube root of 2, but this is an irrational number, and thus it cannot be expressed as a repeating decimal. To estimate the solution to this equation we will develop an algorithm that will give a sequence of estimates to the solution, each of which will be a better estimate than the previous one. The method will use the Intermediate Value Theorem repeatedly applied to the continuous function g(x)=x^3.

First estimate: Consider g(1)=1 and g(2)=2^3 = 8. Since 2 is between 1 and 8, by the Intermediate Value Theorem we have a number c between 1 and 2 where g(c)= c^3 = 2. See Figure ***. Well this is not a great estimate, but it's a start. The solution is a number between 1 and 2.

Second estimate: Find the midpoint of the last estimating interval. This is 1.5. See Figure ***. Now g(1.5)=$(1.5)^3$ = 3.375. Since 2 is between 1 and 3.375, by the Intermediate Value Theorem we have a number c between 1 and 1.5 where g(c)=c^3=2. Well this is a slightly better estimate, but it's still not



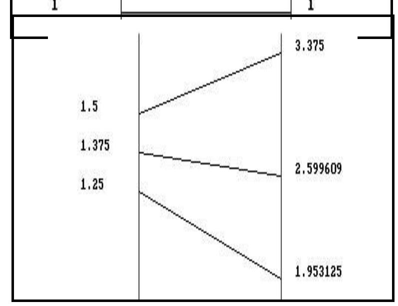


Figure 7

great. The solution is a number between 1 and 1.5.

Third estimate: Find the midpoint of the last estimating interval. See Figure ***.

This is 1.25. Now g(1.25)=$(1.25)^3$ =1.953125. Since 2 is between 1.953125 and 3.375, the Intermediate Value Theorem implies there is a number c between 1.25 and 1.5 where g(c)=c^3=2. Well this is a slightly better estimate, but it's still not great. The solution is a number between 1.25 and 1.5.

The next estimate would be the midpoint of the last interval, namely 1.375. However we'll stop here since the continuation of this process should be clear now. We bisect each successive interval to find a new estimate. See Figure ***. Then we compare it with the last estimates to determine in which of the two resulting pieces to look for the next estimate.

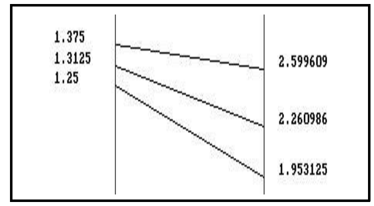


Figure 8

Figure *** shows that the solution is a number between 1.25 and 1.28125. Since at each stage the next estimate is found by bisecting an interval, the resulting estimates can be improved to any desired precision by continuing the process.

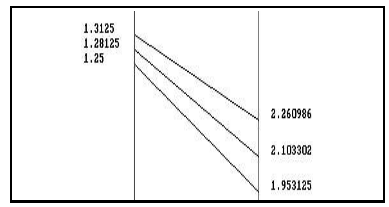


Figure 9

Comment: This method is called the **Bisection method** for estimating the solution to an equation. Usually the equation is transformed into an equation of the form P(x)=0. Though this method relies only on continuity and locating two values of the function P with different signs, it is still one of the most commonly used methods for estimating solutions to equations.

Exercises I.I.2

In problems 1 - 6, use the bisection method four times to estimate the solution of the given equations in the given intervals.

1. $x^2 - 5 = 0$. [2, 3] 2. $x^5 + 2 = 0$. [-2, 0]
3. $x^3 - 7 = 0$. [1, 2] 4. $x^3 - x - 3 = 0$. [1, 2]
5. $2x^3 = 4x - 1$. [0, 1] 6. $x^3 = x + 1$. [1 , 2]
7. Prove that the equation $x^5 - 3x^4 + 2x - 5 = 0$ has at least one real number solution .
8. Prove that the equation $x^7 - 4x^6 + 2x^3 - 35 = 0$ has at least one real number solution .
9. Prove that any polynomial equation of odd degree has at least one real number solution .
10. Prove that the equation $x^6 + x^5 - x^4 - x^3 - x^2 - x + 1 = 0$ has at least one root between 0 and 1.

*11. Write a program for either a computer or a calculator that will use the bisection method repeated N times to estimate the solution to problem 1.

A. Estimate the solution using your program with N = 4, 10, 100.

B. Find N so that when you run your program N times, your estimate will be within .001 of the exact solution. Explain why your solution is correct. Run your program to achieve the estimate with the desired precision.

*12. Modify your program in problem 11 to estimate the solution to problems 2 through 5 within .001 .

13. Suppose g is a continuous function for all real numbers and g(x) = 0 only at x = 0, 5 and 10.

What would be the smallest number of values of g you would need to solve the inequality g(x) > 0?. List all the possible solutions for g(x) > 0 and explain how you would determine which one of these was in fact the solution.

14. A fixed point for a function f is a number c where $f(c)=c$. Suppose that f is a continuous function and $0 \leq f(x) \leq 1$ for all $x \in [0,1]$. [In this exercise you will show that any such function will have a fixed point.]

a) Let $g(x) = f(x) - x$. Explain why g is continuous on [0,1] and **for some c $\in$ [0,1] , g(c)=0**. Draw a figure that illustrates a function f that satisfies the condition and the related function g.

b) Using part a) prove that **for some c $\in$ [0,1] , f(c) = c.**

15. What is wrong with the following argument? Let $f(x) = 1/x$. Then $f(-1) = -1$ and $f(1)=1$. Apply the intermediate value theorem to the number 0. Since $-1 < 0 < 1$, there is a number c with $-1 < c < 1$ where $f(c) = 0$. Therefore $1/c = 0$. Multiply both sides of this equation by c and we have $1 = 0$.

16. **Continuous Random Variables.** We say that a random real variable ***X* is continuous on the range [*a*,b]** if the cumulative distribution function F(A) = the probability that $X \leq A$ is a continuous function on the interval [a,b] with F(a) = 0 and F(b) = 1. The **median of *X*** is a value M where ***a* < M < b and F(M) = 1/2**.

a) Use the Intermediate Value Theorem to explain why any continuous random variable on a range [a,b] has a median.

b) Find the median M for the random variable X when F(A) is given as follows on the range [0,2]. Find the probability density of M.

i) $F(A) = .5 A$.　　ii) $F(A) = 1/4 A^2$.

iii) $F(A) = 1/8 A^3$.　　iv) $F(A) = 1/16 A^4$.

17. Suppose F(A) = sin(A) describes the probability that a random variable Y is less than or equal to A where $0 < A < \pi/2$. Find the median M of Y. What is the point probability density for Y at M.

18. Suppose that an object is inside a sphere of radius 1 and it is equally likely that it is at any point in the sphere. Let R denote the random variable that measures the distance from the object to the center of the sphere. [See Exercise I.C.1.7.] Find the point probability density of the median value of R on [0,1].