I.G. Using the Derivative - Developing Intuitions.[1]

Introduction.

> What is the explanatory role of the derivative in a model? What is the predictive role of the derivative in a model? What is the decision making role of the derivative in a model?

Modeling [VVPBS]: Models can have explanatory and/or predictive powers that help us understand the past and plan for the future. A model can help us date an archaeological find or predict how long the effects of a drug will last. In the example of the trip, our model can help us recall where we were a few hours ago or estimate when we will arrive at our destination. The derivative as a tool for modeling provides information about a point on a curve, about motion at an instant in time, and about the dynamics of a function at a single number. Furthermore, knowledge or assumptions about the derivative at one or several points can tell us about change over an entire interval (for a curve, for time, or for numbers).

When we consider a variable or values of a function on some interval, we are often concerned with changes for either a short interval, a long interval, or a *very* long interval. In the language of modeling, we sometimes describe planning, historical analysis or forecasting with the phrases "short term", "middle range", or "long term". Another way to describe information uses the terms *local* or *relative* (referring to what happens relatively close to a particular point or number), and *global* or *absolute* (indicating the findings concern all meaningful uses for the variable).

In more concrete terms, to describe a test score as the highest or lowest for a group of people in a room at a particular time gives **local** or **relative** information about the test scores. To describe a score on a test as the highest or lowest possible for any administration of that test tells us something about the score that is **global** or **absolute** without consideration of place or time.

In this section, we begin examining applications of the derivative – both local and global – for explanation (I.G.1) and for prediction or extrapolation (I.G.2). Our approach at this stage will be informal, relying on your ability to recognize patterns and generalize, rather than trying to be rigorous. Our purpose is to develop intuitions through experience. We will treat these and other applications more carefully in Chapters III, IV, and V.

Read the examples here carefully and, as you read, note connections between function qualities, characteristics of the derivative, and the interpretations we have developed. These connections will provide a background for making sense of function qualities and derivatives through the remainder of our work.

I.G.1. The First Derivative - Indicating Change.

The velocity of a moving object gives us information about how the object's position is changing, and, *vice versa*, information about how a moving object's position is changing informs us about its velocity. Similarly, the slope of a line gives us information about the appearance of the line and *vice versa*. Likewise, the marginal profit gives us information about

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[1] (This section may be deferred for treatment after Chapter II)

how profit is changing and *vice versa*. In this section, we begin to study the correspondence between functions and their derivatives more carefully.

Example I.G.1: Consider the linear function $l(x) = 5x\text{-}3$ in each of the interpretations: motion, graphic, and economic.

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Motion: We can interpret this function dynamically, giving the position of a person jogging at a velocity of 5 meter per second from an initial position located 3 meters before the starting line on the running course. See Figure 1.

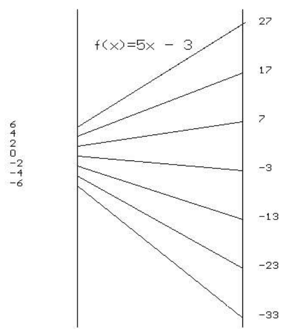

Figure 1

As time progresses the runner's position increases relative to the starting line.

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Graph: We can interpret this function graphically as a line with slope 5 and Y-intercept (0,-3). See Figure 2.

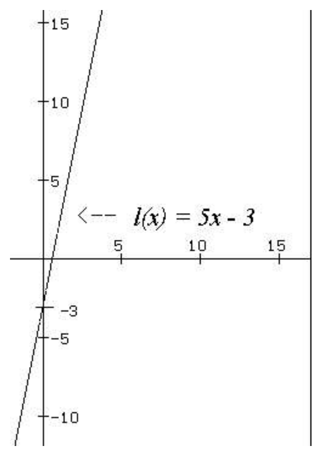

Figure 2

As we scan the figure from left to right we see the line rising on the graph.

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Economics: We can interpret this function economically, giving the profit for a small enterprise with a marginal profit of \$5 per unit and an initial loss of \$3 when no units are produced.

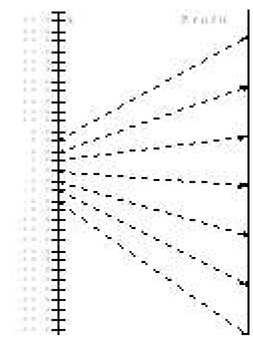

Figure 3

With more production, the profit increases.

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We have already noted the connection between these interpretations in Section I.C, but our focus now is on the fact that **for any x we have $l\,'(x)=5$.**

Key Question: What does a positive derivative tell us about l and its interpretations?

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In the mapping figure visualizing the jogger, we see that as the point on the source line (time) moves up (indicating a later time), the corresponding point on the target line also moves up (indicating a larger value for the position function).

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On the graph we understand from the positive slope that as we scan from left to right, the line moves up.

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In the mapping figure visualizing the profit function, we see that as the point on the source line (production level) moves up (indicating greater production), the corresponding point on the target line also moves up (indicating greater profit).

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The word *increasing* is commonly used to describe these situations and this kind of function behavior.

More formally, we say a function f is **increasing on a set** or domain S if the values of f are larger for larger numbers in S. We consolidate this concept in a more technically-phrased definition as follows:

Definition: A function f is **increasing on a set** or domain S, if, for any a and b in S, $a < b$ implies $\boldsymbol{f(a) < f(b)}$.

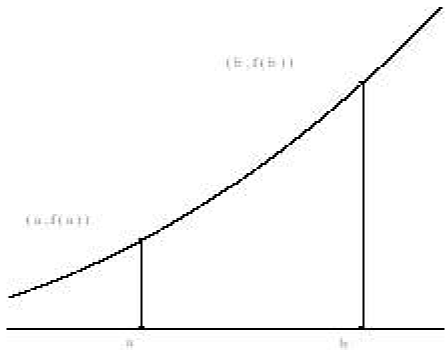


Figure 4
Graph for an increasing function.

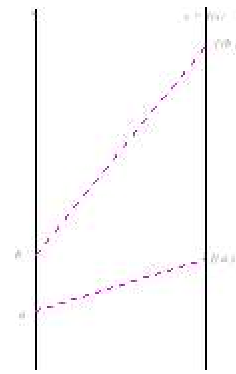


Figure 5
Mapping diagram for an increasing function.

Example (continued): Verify that the linear function $l(x) = 5x-3$ is increasing on its domain in the technical sense just defined.

Proof: Suppose a and b are real numbers with $a < b$.

Then $5a < 5b$ and
so $l(a) = 5a - 3 < 5b - 3 = l(b)$.

Thus for any numbers a and b where $a < b$, we have shown that $l(a) < l(b)$, completing the argument that l is an increasing function. **EOP.**

Comment: The function l is increasing for all real numbers, making this a **global property** of l.

You can consider a similar linear example where the derivative (velocity, slope, marginal profit) is negative to see that the appropriate descriptive term for that situation is *decreasing*. Here is the technical definition for a decreasing function:

Definition: A function f is **decreasing on a set** or domain S, if for any a and b in S, $a < b$ implies $\boldsymbol{f(a) > f(b)}$.

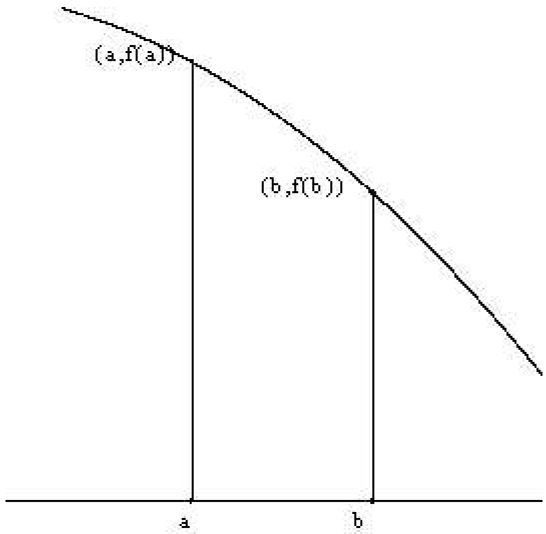


Figure 6 Graph for a decreasing function.

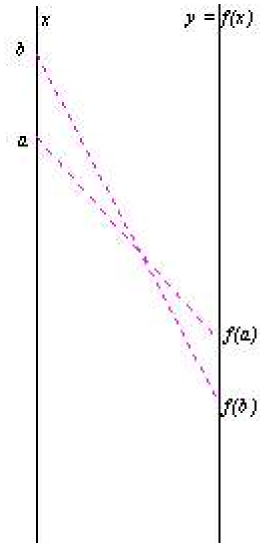
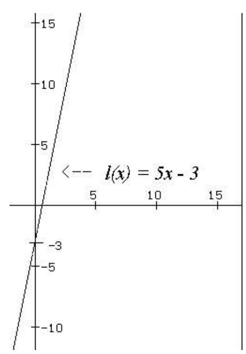
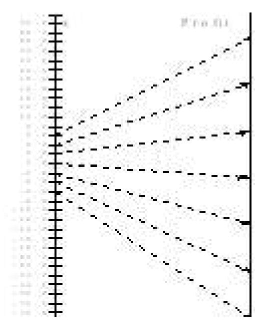
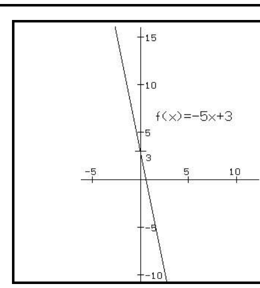
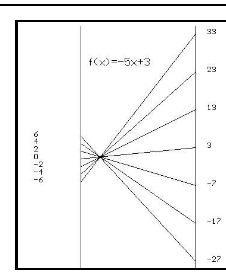


Figure 7 Mapping diagram for a decreasing function.

Notice that the only difference in the technical definitions of increasing and decreasing is in the relation of $f\,(a)$ to $f\,(b)$. This should make sense in both the mapping figures and the graphical interpretations using linear functions as examples.

Linear Functions	Graph	Mapping Diagram
Increasing	 **Figure 8**	 **Figure 9**
Decreasing	 **Figure 10**	 **Figure 11**

Example I.G.2: Let's consider the quadratic function q where $q(x) = -x^2 + 6x - 5$ and its variation in relation to its derivative. The graph of q is a parabolic curve with vertex at (3, 4). See Figure 12.

As x goes from left to right the curve moves up for $x < 3$, while for $x > 3$ the curve moves down. The mapping diagram (See Figures 13 and 14) of q helps visualize a motion interpretation, showing the position of a moving object increases *before* time 3 while its position decreases *after* that same instant. Alternatively the mapping diagram (See Figures 13 and 14) of q can visualize a profit interpretation showing the profit increases when producing *less than* 3 units while it decreases when producing *more than* 3 units.

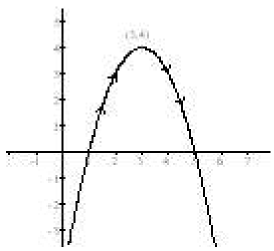


Figure 12
Graph of
$q(x) = -x^2 + 6x - 5$

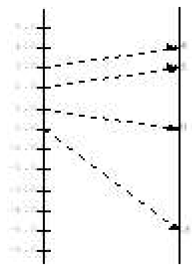


Figure 13
Mapping diagram of q
with $x < 3$.

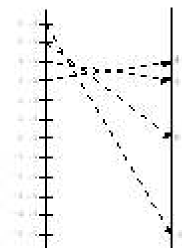


Figure 14
Mapping diagram of q
with $x > 3$.

The increasing or decreasing quality of this function depends on the interval considered, making it a local property. We describe this function as increasing for $x < 3$ and decreasing for $x > 3$. It is often convenient to describe these local properties using interval notation for these domains. So we say that q is increasing on the interval $(-\infty, 3)$ and decreasing on the interval $(3,\infty)$.

Now let's compare the information about the function increasing and decreasing with information about the derivative of q for the corresponding intervals.

We observe first that $q'(x) = -2x + 6$.

Motion Interpretation: In the context of a moving object with position given by the function *q*, we interpret the derivative of *q*, *q'*, as the instantaneous velocity of the object. For the example with position $q(x)$ at time *x* we have that the velocity is given by $q'(x) = -2x + 6$, resulting in positive velocities *before* time 3 and negative velocities *after* time 3. This matches our previous observation that the object's position is increasing until time 3 after which the position is decreasing. At the instant 3 the object's instantaneous velocity is 0, so the object "stops" for that instant while at any other instant it is moving. At time 3, when the direction of the object's motion changes, the object reaches its highest position, 4, the *maximum* value for *q*.	**Graphic Interpretation**: In the context of the graph of the function *q*, we interpret the derivative of *q*, *q'*, as the slope of the line tangent to the graph. For the example we have that the slope of the tangent line is given by $q'(x) = -2x+ 6$, resulting in positive slopes when *x* is less than 3 and negative slopes when x is greater than 3. This matches our previous observation that as *x* goes from left to right the curve moves up for $x<3$, while for $x > 3$ the curve moves down. You might have expected this from the *graphical interpretation of q alone* based on tangent lines for points on the graph to the left and right of the vertex at (3,4) where the graph has its highest position. We also see that at the vertex, $q'(3) = 0$, so the tangent line at the vertex is horizontal.	**Economic Interpretation:** In the context of economics with profit given by the function *q*, we interpret the derivative of *q*, *q'*, as the marginal profit. For the example with profit $q(x)$ when producing *x* units, we have that the marginal profit is given by $q'(x) = -2x+ 6$, resulting in positive marginal profit when producing *less than* 3 units and negative marginal profit (*i.e.* marginal loss) when producing *more than* 3 units. The mapping figure visualizations show that when the production level is below 3 units the profits increase for increased production. When production goes over 3 units the profits decrease with further increases of production. At 3 units the profits are at their highest, 4. This information can be useful for making business decisions. [2]

Example I.G.3: (**Probability Interpretation.**) Consider an experiment in which we measure a random variable R. Suppose the cumulative distribution function, F, for R is a linear function on the interval [0,2]. (See Section I.C.1 for a review of probability distribution and density functions.) Thus F(0) = 0 and F(2) = 1, so F(A) = ½ A. Whenever $0 \le A < B \le 2$, the probability that R<A is less than the probability that R<B, *i.e.*, F(A) < F(B), so the function F is increasing for the interval [0,2]. [Of course this is “easy” to see because F is linear and has a positive slope.]

For a nonlinear example, suppose the cumulative distribution function, F, for R is a differentiable function on the interval [0,2] represented by the graph given in Figure 15. Consider A and B in the interval [0,2]. Whenever $0 \le A < B \le 2$ the probability that R<A is less than or equal to the probability that R<B, *i.e.*, $F(A) \le F(B)$, so the function F is increasing for the interval [0,2].

2. How would this information influence decisions? As a person making decisions about production levels you might be reticent to increase production at a level above 3. With local information only of increasing profits at production levels below 3 you would not see how profits will eventually turn around with higher production levels and actually decrease steadily for production levels over 3 units.

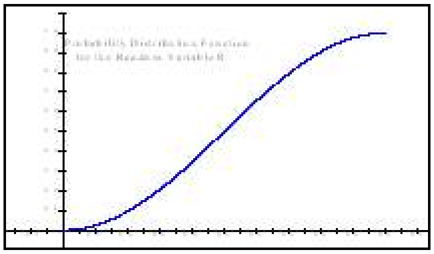


Figure 15 Distribution Function F for the Random Variable R

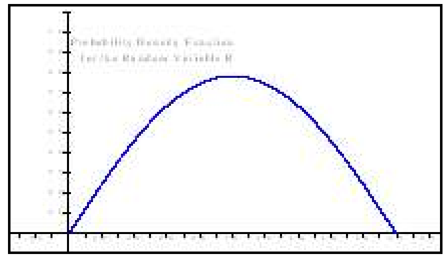


Figure 16 Density Function F' for the Random Variable R

The derivative of F, shown in Figure 16, is the density function of R. As you might expect, the density function is non-negative for all *x* in the interval [0,2].

Note: In Figure 16, where A=1 the derivative has its greatest value. Intervals around A=1 have relatively high density while the graph of F in Figure 15 is steeper for the same intervals. On the other hand, for intervals around A = 0 and A = 2, in Figure 16 we observe lower density while the graph in Figure 15 is flatter.

In the context of the probability interpretation of the derivative, intervals with higher probability density, i.e., with higher derivative values, have greater likelihood of the value of the random variable occurring in those intervals. These are also the intervals where the distribution function's graph is steeper.

We can use mapping diagrams here as well to explore the relationship of the probability density- distribution connection with the derivative. See Figures 17 and 18. Examining the distribution mapping figure (Fig.17) we see that the distribution values are increasing. More interesting perhaps is the feature that F(1)=0.5.

In Figure 17, although the source intervals correspond with equal sizes, their related intervals in the target of the distribution function differ in size. When the interval for the distribution function's values is longer, the probability for the random variable falling in that interval is greater, as is evidenced by a higher density function value in its mapping diagram (Figs 17 and 18). When the interval for the distribution function's values is shorter, the probability for the random variable falling in that interval is less, as is evidenced by a lower density function value in its mapping diagram (Figs 17 and 18).

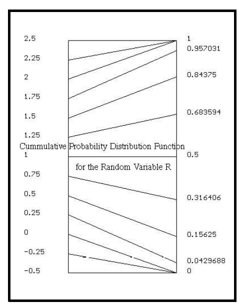


Figure 17
Mapping Diagram for F

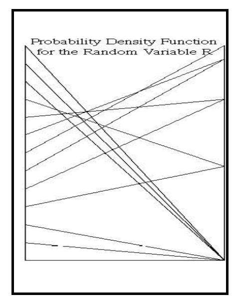


Figure 18
Mapping Diagram for F'

In these examples and their interpretations (motion, graphical, economic, and probability), we have noted some connections between the sign of the derivative (positive or negative) and changes in the controlled variable (increasing or decreasing). These connections will be investigated more thoroughly in Chapter III.B.

I.G.2. Reversing information:
How information about rates leads to information about primitives.

The Derivative and The Flow of Information: In our motivation for the derivative we used information about position to estimate velocity, about points to estimate slope, about costs and profits to estimate marginal costs and profits, and about probability distributions to estimate probability density. For a function's derivative at a particular number, we estimated by considering quotients determined by values of the function for numbers close to the given number.

What is quite remarkable, yet sensible, is that we can reverse the flow of information. From knowledge about our position and velocity at one time, we can estimate our position a little earlier or later. From knowing a point on a curve and the slope of the tangent line there, we can estimate where nearby points on the curve are. Knowing the marginal cost and profit at one production level can help us estimate the costs and profits for small changes in production levels. And knowing the probability density of a number allows us to estimate the probability that a random variable will assume a value relatively close to that number. All these examples show the merit of knowing values for a function and its derivative at a single point to estimate the value of the function at nearby points.

As we have seen in Section I.G.1, the derivative can provide qualitative information about a function, *e.g.*, a positive derivative indicates an increasing function.There are many ways to use information about the derivative to reconstruct the primitive function values or estimate them as needed. Information about the derivative might include the numerical value of the derivative at particular numbers, or merely how the value of the derivative can be determined. We may be given much information about the derivative, but know the value of the function at only one point.

The examples in this section illustrate the reversed flow of information. More examples of using derivative information to learn about primitive functions will appear throughout the remainder of the text. This is one of the most important themes of the calculus.

> The word "primitive" has a connotation in common language of being early and initial, in contrast to the term "derivative" which connotes something which arises later and after some processing. These words seem quite appropriate to describe the relation of a function f to the derivative function f' that measures the primitive function's rate of change.

In our interpretations of the derivative, we want to recover information about primitive measurements from secondary, derived measurements. More specifically with a motion interpretation, we want to recover information about a car's position on a trip from knowing the car's velocity at various (or perhaps all) times and the car's position for at least one moment. For the graphical interpretation, we want to recover the graph of a function from knowing only the slope of the tangent line at various points and at least one point on the graph. For an economics interpretation, we want to reconstruct the primitive profit (or cost or revenue) function based on information about marginal rates at several levels of production and knowledge of the profit (or cost or revenue) at one level of production .

Let's consider more detailed examples, first with an algebraic example and then with numerical and graphical examples afterward.

Example I.G.4: (Algebraic) Suppose $f'(x) = 8x - 3$ for all x and $f(2) = 6$.
Find $f(0)$ and $f(10)$.

Discussion: We would like a simple algebraic characterization of the function f that will allow us to compute $f(0)$ and $f(10)$. Of course there are an enormous number of functions that have their value being 6 at 2, *i.e.*, with $f(2) = 6$. The condition specifying the derivative, $f'(x) = 8x - 3$, diminishes the number of possibilities considerably. [We will see in Chapter IV that there is **only one possible solution** to such a problem. This is not hard to understand when you use the motion interpretation. [Perry Mason] Consider two objects being at the same place at time 2 (on different days) and both traveling at equal velocity at any time. Then these objects will always have the same position function , thus **only one position function is possible**.]

Our initial strategy to solve this problem is to **guess** a function that might work **and then check** whether our guess is correct. We'll name our guessed function g, so after the guess we will check whether $g'(x) = 8x-3$ and $g(2) = 6$.

Solution: Let's guess $g(x) = 4x^2 - 3x$. You can check that $g'(x)= 8x - 3$, which is some success, but unfortunately $g(2) = 10$, not 6. So this candidate function ***g* is not the function we desire** to satisfy both requirements for the example.

Since our initial guess has $g(2) = 10$, we can adjust our result by subtracting 4 from the value of $g(2)$ to obtain 6.

This leads us to revise our guess with the function r where $r(x)= 4x^2 - 3x -4$.
This is the solution we are seeking. You can verify for yourself that $r'(x) = 8x - 3$ for all x and $r(2) = 6$.

Now that we have discovered the correct algebraic characterization of f, namely,

$$f(x) = 4x^2 - 3x - 4,$$

we find $f(0) = -4$ and $f(10) = 366$.

Let's look at the numerical side of this type of problem. If we are using only numerical information at a few points, predictions about other points will be, at best, estimates based on assumptions about how given information relates to changing variable values. These assumptions determine a mathematical model, our basis for further analysis, discussion, and inference.

Example I.G.5: (Numerical) Let's consider some numerical data that is at least consistent with the last algebraic example at $x = 2$. Suppose $f(2) = 6$ and $f'(2)=13$.
Based on this information ***alone***, how can we make an estimate for $f(0)$ and $f(10)$?
Discussion: With this little information, a sensible approach to making an estimate of $f(0)$ is to assume that the derivative of f is estimated by the difference quotient $\frac{f(2)-f(0)}{2-0}$.

Solution: Using the given hypothetical data about f gives $\frac{f(2)-f(0)}{2-0}=\frac{6-f(0)}{2}\approx f'(2)=13$, *i.e*, $\frac{6-f(0)}{2}\approx 13$. Now using the algebra of approximation we multiply the estimate by 2 giving the new estimate $6-f(0)\approx 2\cdot 13 = 26$, and so we can approximate $f(0)\approx$ 6 - 26 = -20.

Following the same approach to estimate $f(10)$, we estimate the derivative of f at 2 with $\frac{f(2)-f(10)}{2-10}$. Using the given data gives $\frac{f(2)-f(10)}{2-10}=\frac{6-f(10)}{-8}\approx f'(2)=13$. Now the algebra of approximation suggests that $6-f(10)\approx -8\cdot 13 = -104$. So we estimate $f(10)\approx$ 6+104 = 110.

Comment: Let's compare the results from our previous algebraic example with these numerical estimates. From the much stronger assumptions of the algebraic example we found the values exactly: $f(0)$ = -4 and $f(10)$ = 366. With the weaker assumptions made for our numerical estimates we arrive at estimates $f(0)\approx$ -20 and $f(10)\approx$ 110.

Notice that **at *x* = 2** the two examples have consistent information about the value of f and the value of the derivative f'. The different results make sense because of the difference in assumptions in the two approaches. **With complete knowledge** of the values of the derivative of f for all x, $f'(x)$, we are able to reconstruct the function f exactly. In contrast **with the very limited knowledge** of the value of the derivative **at 2 only,** we could assume that estimates of the derivative could be used to estimate the values of the function- reversing the flow of information.

Where previously we used information about the values of the function to estimate the derivative at a particular point, now we use information about the derivative at a particular point to estimate values of the function. **EOC**

A Motion Interpretation for Example I.G.5: We can use the motion interpretation with units of meters and seconds to make some sense of the estimates obtained in Example I.G.5. Treating the derivative as the velocity of a moving object, we interpret $f'(2)=13$ as a statement that at time 2 seconds the object has a velocity of 13 meters per second. To estimate the position at time 0, we recognize that this is 2 seconds earlier and make the assumption that the object was traveling at a constant velocity. So, during the time between 0 and 2 seconds the object would move 13·2 = 26 meters. Using 26 meters as an estimate of the change in position we estimate the position at 0 seconds to be 6-26= -20 meters. Similar thinking leads to the estimate at time 10.

The quality of the estimates should make sense from the motion interpretation. With a longer time involved in the estimation and without knowing more about the context, there is less likelihood that the assumption of constant velocity will be sensible. So starting from information of the position at time 2 seconds, we would expect the estimate for the position at 0 seconds to be more accurate than the estimate for the position at 10 seconds.

Note: You might think that Example I.G.5 is far from correct in its estimates in comparison to Example I.G.4. It is important to emphasize again that the assumptions for these examples were not the same, even though the information at 2 is consistent. In fact we know very little about the function's derivative in example I.G.5, so we don't know that the function in this example is anything like the function in I.G.4 except close to 2, where the difference quotients that estimate the derivative at 2 are close to 13 for both examples.

We turn to a graphical example for reversing the flow of information. In a way this visual approach may be the most elementary interpretation. We will see that the idea for this reversal is so sensible that it could be used as an assignment for a primary school student.[3]

Example I.G.6. Figure 19 is called a **direction (or slope) field** and is a visualization of the information in the derivative equation $f'(x) = 8x - 3$. In this figure, we have a section of the coordinate plane with short line segments of differing slopes drawn using the following procedure: At a point with coordinates (a,b) the line segment is centered at that point with slope $8a - 3$. For example, the slope of the line segment at the point (1,2) is equal to the value of the derivative at that point, *i.e*, the slope is $f'(1) = 8\bullet 1 - 3 = 5$. In fact, the slope of each line segment at a point $(1,b)$ is the same since it is also equal to the value of the derivative when $x = 1$, *i.e.*, the slope is $f'(1) = 5$.

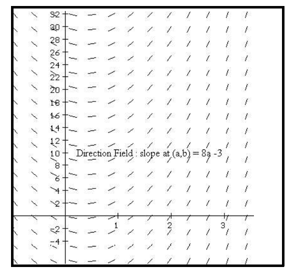


Figure 19
Direction Field for $f'(x) = 8x - 3$.

The geometric problem comparable to I.G.4 is to find a curve that passes through the point (2,6) so that its tangent lines conform to the pattern of the line segments of the direction field in the figure.

Discussion: In other words, we want to draw a curve through (2,6) so that the tangent line for any point on the curve would be consistent with the rule given for drawing segments in the direction field. In some ways, this can be restated as a problem in recognizing and integrating the visual pattern in the field to draw a single curve.

Solution: Figure 20 shows the solution curve (called an **integral curve**) for the given slope field, drawn with the aid of a computer. You should also try this as a free hand sketch to get a sense of how the direction field controls the shape of the curve.

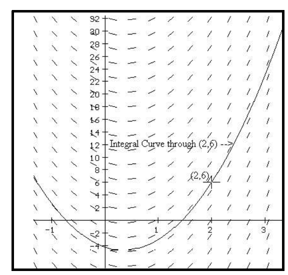


Figure 20

The graphical problem of estimating the value of $f(0)$ is to estimate the second coordinate of the point on the curve with first coordinate 0, that is, the Y- intercept of the curve. Based on the sketch in Figure 20 it would appear that point has coordinates (0,-4). We will pursue this visual interpretation of the reversal problem further in Chapter IV.

Exercises I.G: Work in progress.

1. **Each of the following figures is the graph of a function. For these functions describe the intervals where you believe the derivative functions will be positive and those where the derivative functions will be negative. Discuss where the derivative functions are 0 and how this relates to the graphs.**

a.

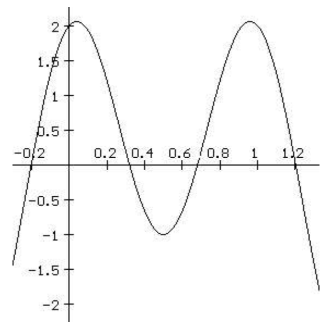


Figure 23

b.

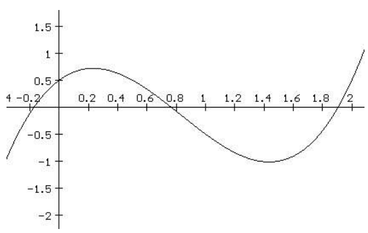


Figure 24

c.

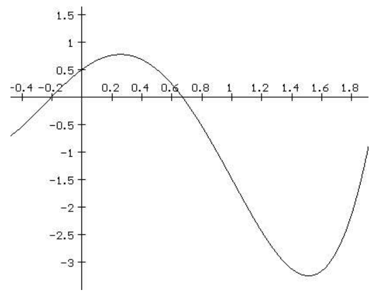


Figure 21

d.

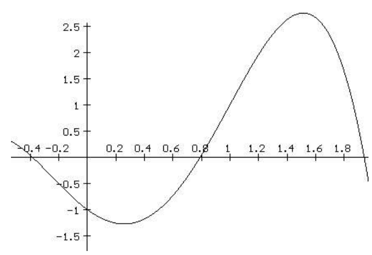


Figure 22