Chapter 0: Introduction, Background, and Other Goodies
A. What is Calculus?

Sensible Calculus © 2008

There are all too many books with titles using the term "Calculus" and courses with the same title to think that this is the first time you have seen this word. Hey, you're in a calculus course right now! But do you know why the subject is called "calculus" and precisely what it is? The answers are not simple. To begin, we'll relate our understanding of the background of calculus before the term "calculus" was identified as a specific subject area of mathematics.

What is "calculus"?

Quite simply a "calculus" is a method for systematically determining a result, for arriving at a conclusion, or (if you don't mind the redundancy) for calculating an answer[1]. [Add note/box saying something linguistic about stones, hard materials, and dentistry.] In this sense there are many calculi, such as the calculus of propositional logic, the calculus of set operations, the calculus of probabilities, *etc.* But when someone talks about "**The Calculus**," be it "differential" calculus, "integral" calculus, or the calculus of infinite series, the reference is usually to "**The Calculus of Isaac Newton** (1642-1727) **and Gottfried Leibniz** (1646-1716)."

This calculus provides procedures for *solving problems in the analysis of change*:

- determining rates of change,
- predicting the amount of change and
- explaining the quality of change, and
- connecting the concepts of change with the language and symbolism of algebra that describes change.

This calculus also develops tools for *solving problems of geometry:*

- determining a line tangent to a curve or finding the area of a planar region,
- predicting the shape and explaining the graphic qualities of a curve, and
- connecting these geometric concepts to the language of algebra that describes geometry.

Before Newton and Leibniz, several great mathematicians had studied many of the same physical and geometrical problems as those treated by the calculus, from Euclid and Archimedes in antiquity to Descartes, Fermat, and Pascal in the early 17th century. Newton and Leibniz differed from those who had worked on these questions before them by achieving a general overview to the problems. Their approaches, developed independently, solved the problems **using systematic techniques of calculation** that depended fundamentally on **algebraic descriptions of the problems**. With these new techniques a user could avoid repetitious conceptual analysis of each different

---

[1] From the Complete Reference Library Dictionary, Mindscape. The American Heritage Dictionary of the English Language. 3rd Edition, 1980, Houghton Mifflin.
Calculus a . The branch of mathematics that deals with limits and the differentiation and integration of functions of one or more variables.
b . A method of analysis or calculation using a special symbolic notation.
c . The combined mathematics of differential calculus and integral calculus. A system or method of calculation.[Latin, small stone used in reckoning]

curve or formula. The conceptual analysis was summarized in the justifications of the calculus rules, which were the primary achievement of the new calculus. While the *results* for actual problems contained within the works of Newton and Leibniz may not have been new, it was *the methods* that were revolutionary in their generality.

Since its early development, the calculus has grown more important. Its analysis has been applied in many contexts: the physical sciences and engineering, the life sciences, economics, and probability. In fact, the calculus has uses in practically any area of study where change is important. It provides a theoretical basis as well as a practical tool for finding exact or estimated solutions to problems in almost every scientific discipline.

Let's touch briefly on two problems of historical importance to the development of the calculus, the **tangent** problem and the **area** problem.

The Tangent Problem: You may recall a geometric construction, known at least since the time of Euclid (about 300 B.C.E.) for drawing **a line that touches a circle only at one specific point, P, on the circle.** [See Figure 1.] This line is called the **tangent line** to the circle at P. The construction of this line involves two simple steps: i) draw a radius from the center of the circle, O, to the point P; and ii) construct a line at P that is perpendicular to the radius OP. This perpendicular line is the desired tangent line.

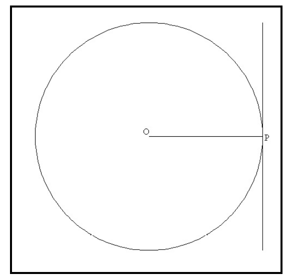

Figure 1

A similar problem for a parabola was solved by Archimedes. (c. 287 B.C.E.–212 B.C.E.) The problem is to **construct a line that touches a parabola only at one specific point, P, on the parabola.** [See Figure 2.] As we proceed in this and the next chapter we will develop methods to solve this problem. This line is called the **tangent line** to the parabola at P.

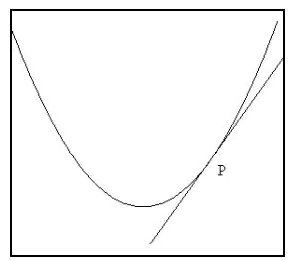

Figure 2

The **general tangent problem** is more difficult to state because the quality of being tangent in general requires a more subtle characterization. Ignoring the issue of what "tangent" means for now, we describe the general tangent problem as the problem of **finding a method for drawing a line tangent to a curve at a specific point P on the curve.**

The Area Problem: The **area** problem is perhaps more familiar today as a measurement problem. No doubt you have learned many formulae for finding the area of common planar regions such as those enclosed by squares, rectangles, triangles, trapezoids, and circles. Ancient treatments of area formulated the problems in terms of providing a geometric construction.

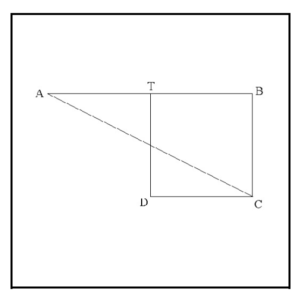

Figure 3

For example, the ancient area problem for a right triangle was to construct a rectangular region with area equal to that of the given right triangle. [See Figure 3.] The geometric solution follows two steps: i) Bisect one leg of the right triangle. ii) Form a rectangle with one of the resulting segments and the other leg. This rectangle has the desired property, i.e., the area of this rectangle is equal to the area of the original right triangle.

The statement of the **general area problem in geometry** was to construct a rectangle with an area that is the same as the area of a given region in the plane.

The connection between geometry, numbers, and algebra is generally considered one of the major contributions of the French mathematician, scientist, and philosopher, René Descartes (1596–1650), though some of the ideas appeared in earlier works of Oresme, Viete, and Galileo.[pronunciation -historical note?]

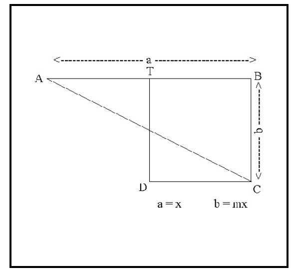

Figure 4

Once measurement and algebra are added to the tools with which we analyze the area problem, we arrive at a familiar formulation of the right triangle area result. If the legs of the right triangle have lengths *a* and *b* then the area *A* of the triangle (and the rectangle) is 1/2 *a b*. [See Figure 4 .] If we use *x* instead of *a* for the length of one leg and *mx* instead of *b* for the length of the other leg then we have the area determined by the numbers *m* and *x* where *m* is the ratio of *b* to *a, m* = *b/a.*

$$A = \frac{1}{2}x \cdot mx = \frac{m x^2}{2} \quad .$$

The Fundamental Theorem of Calculus.

One of the most important results of the calculus, now referred to as the **"Fundamental Theorem of Calculus,"** relates the tangent problem to the area problem. This result was known to Isaac Barrow, Newton's teacher and predecessor as professor of mathematics at Cambridge University in England. In its geometric formulation we refer to it as **Barrow's Theorem**. Here is a slight paraphrasing of Barrow's original result.

Barrow's Theorem

Hypotheses:

Suppose Y is a curve intersecting the line OX only at the point O.

Suppose further that the curve A has the following properties:

- For a point P chosen on the curve A there is a corresponding point Q on the curve Y so that the segment PQ intersects the line OX at the point R forming a right angle.
- The numerical value of the length of segment PR is the same as the numerical value of the area of the region enclosed by the curve Y, the segment OR and segment RQ. (See Figure 5 and note the shading of the appropriate region.)

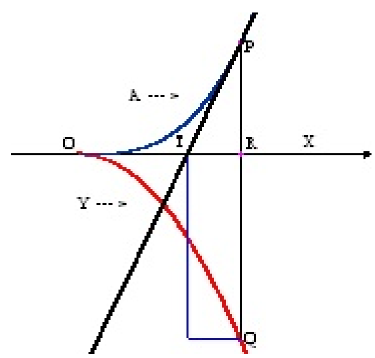

Figure 5

Suppose the point T is chosen on the segment OR so that the numerical value of the rectangle with sides TR and RQ is equal to the numerical value of the length of the segment PR.

Conclusion: The line TP is tangent to the curve A at the point P.[2]

---

[2] A proof of Barrow's Theorem based on his original argument appears in the Appendix at the end of this section.

Comments: Barrow's Theorem certainly relates the tangent and area problems. To solve the tangent problem for the area curve A, we need only solve the area problem related to the curve Y, that is to find the point T so that the rectangle determined by T has the appropriate area.

Though in Figure 5 the curve Y appears to be a line, the curve Y can be any curve, without specifying any special type. Of course with a more general curve for Y the problem of finding the curve A precisely becomes much more difficult. **At this stage don't worry about the scales in Figure 5. With the appropriate units the figure is accurate, but that is not our concern at this stage.** Our purpose here is to show a coonection between the area and tangent problems.

We can state Barrow's Theorem in slightly more modern terms, using the concept of the slope of a line. If the line tangent to A at the point P has a slope, $m_{\tan}$, then Barrow's Theorem says this slope is precisely the value of the length of the segment RQ.

$$m_{\tan} = \frac{PR}{TR} = \frac{TR \cdot RQ}{TR} = RQ$$

What is most notable about Barrow's Theorem is the absence of algebra from its original statement and proof. It was primarily the geometric content that appeared in Barrow's work. Only when Newton and Leibniz arrived at algebraic versions of this result did its utility become apparent.

Newton and Leibniz both developed algebraic methods for a calculus of tangents and with that they were able to apply Barrow's Theorem to explore algebraic methods for connecting the calculus of tangents to a calculus that systematically resolves many problems of area, as well as problems of volume, arc length, etc.

What then is the calculus?

Briefly again, calculus is a **conceptual framework which provides systematic techniques for solving problems suitably posed in the language of analytic geometry and algebra**.

Is it easy to list the types of problems that calculus can solve?

Some problem types frequently mentioned in discussions of the calculus over the centuries of its development are quite easy to characterize. Other problem types have developed in the past century or so as a result of the expansion of the use of mathematics (beyond the physical sciences of physics, chemistry, and engineering, into disciplines such as biology, medicine, and economics). More recently,with the increased importance of probability and statistics in all of the sciences, further applications of the calculus have arisen .

Here is a list of some traditional and non-traditional problems that we will investigate and solve as we progress in our study of the calculus.

1. **The Tangent Problem.** Determine the line tangent to a given curve at a given point. (Also, define precisely the concept of "tangent.")

For example, determine the line in the plane tangent to the circle with equation $X^2 + Y^2 = 25$ at the point (-4,3).

2. **The Velocity Problem.** Determine the instantaneous velocity of a moving object. (Also,

define precisely the concept of "instantaneous velocity.")

For example, determine the instantaneous velocity at time t = 5 seconds of an object moving on a straight line with its distance from a given point P at time t seconds being $t^2 + 6t$ meters?

3. **The Extremum Problems.** Determine the maximum and minimum values of a dependent variable.

For example, when $Y = X^2 - 6X$, determine any maximum and minimum values for the dependent variable Y as X is varies over real numbers between 0 and 10.

4. **The Tangent-Curve Problem Reversed.** Determine a curve satisfying two conditions:

i) The curve passes through a given point, and

ii) at any point on the curve, the tangent at that point fits a specific description based on the point's location.

For example, determine a curve through the point (1,2) so that the slope of the tangent to the curve at the point (a,b) is $2a - b$.

5. **The Position Problem. (The Velocity Problem Reversed.)** Determine the position of an object moving on a straight line at a given instant, from knowledge of two specifications:

i) Its initial position, and

ii) its instantaneous velocity at every instant.

For example, determine the position of an object moving on a straight line at time $t = 5$ seconds, knowing its initial position is P on the line and its instantaneous velocity at time t is precisely $t^2 - 6t$ meters per second.

6. **The Growth/Decay Problem.** Determine the size of a population at a given time (past or future) from knowledge of the following:

i) The size of the population at a specific time, and

ii) the rate of growth /decay of the population at any time.

For example, determine a biomass of a population ten hours after an initial observation that the biomass was 950 kilograms and the population is growing at a rate that is proportional to its current size and that after one hour its biomass was 1000 kilograms.

7. **The Area Problem.** Determine the area of a region enclosed by suitably defined curves.

For example, determine the area of the planar region enclosed by the **X - axis,** the lines **X = 2, X = 5,** and the parabola with equation $\mathbf{Y = X^2 - 6X}$.

8. **The Arc Length Problem.** Determine the length of a suitably defined curve.

For example, determine the length of the parabola with equation $Y = X^2 - 6X$ between the points (0,0) and (6,0).

9. **The Probability Expected Value - Mean Problem.** Determine the expected value of a measurement, X, taken during an experiment where we know the probability that $X < A$. [The expected value of X is the number that in theory would be close to any average of the values for X if the experiment is repeated a very large number of times.]

For example, suppose we throw a dart at a circular magnetic dart board of radius 2 feet and we measure X as the distance from where the dart lands to the center of the board. Given that the probability that $X < A$ is $A^2/4$ for $0 \le A \le 2$, what is the expected value of X? that is, what is the expected distance from where the dart lands to the center of the target?

Backgrounds for studying the Calculus: Now that you have a better idea of what mathematical problems we can solve using calculus, we will review some useful background knowledge and skills from your prior studies. The concept and terminology of functions provide important foundations and valuable language for the study of calculus.

To understand calculus you should have a background that includes

1. numbers and variables in the context of algebra;
2. equations and functions both algebraically and visually; and
3. "real world" applications that use functions to relate the quantities involved.

So what is a sensible approach in preparation for the study of calculus?
To Prepare you, our sensible calculus approach will:

1. Review and renew your understanding of numbers and variables as used in algebra.
2. Review and renew your understanding of equations both algebraically and visually.
3. Review, renew, and expand your understanding of functions both algebraically and visually.
4. Connect "real world" applications to equations and functions.
5. Introduce some problem types encountered in calculus where non-calculus techniques can be used to find solutions.
6. Introduce methods from current technology that make analysis easier and that form a foundation for later use of technology in studying calculus.

0.A. Exercises. In these problems assume Barrow's Theorem is true. The problems ask you to find the slope of lines tangent to parabolas arising from quadratic functions of the form $Y = CX^2$. [Now is good time to reread the statement of Barrow's Theorem.]

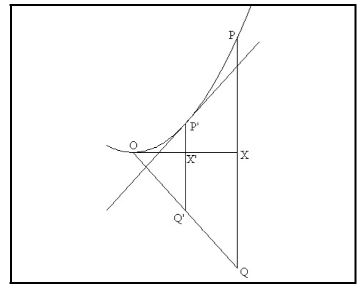

Figure 6
Not drawn to scale.

1. Suppose the right triangle OXQ as given in Figure 6 has side OX of length 5 on the horizontal axis and vertical side XQ of length 10. Above triangle OXQ is the area curve A, which is related to the line Y determined by vertices O and Q.

a. Let *x, q* and *p* be defined as follows:
- *x* denotes the length of segment OX',
- *q* denotes the length of segment X'Q', and
- *p* denoted the length of segment X'P'= *p*

Give an equation relating *q* and *x*. Give an equation relating *p* and *x*. [Remember the numerical value of the length of X'P' is the same as the numerical value of the triangular area of the region enclosed by the X-axis, the line segment X'Q' and the line segment OQ'.]

b. When $x = 3$ find the position of T so that P' T is a line tangent to the curve A at P'.

c. Find the slope of the line tangent to the curve A at P' when $x=3$.

d. Find the slope of the line tangent to the curve A at P' when $x = 1$, 2, and 4.

e. Find the slope of the line tangent to the curve A at P' when $x = a$.

2. Suppose the right triangle OXQ as given in Figure 6 has side OX of length 10 on the horizontal axis and vertical side XQ of length 5. Above triangle OXQ is the area curve A, which is related to the line Y determined by vertices O and Q.

a. Let *x, q* and *p* be defined as follows:
- *x* denotes the length of segment OX',
- *q* denotes the length of segment X'Q', and
- *p* denoted the length of segment X'P'= *p*

Give an equation relating *q* and *x*. Give an equation relating *p* and *x*. [Remember the numerical value of the length of X'P' is the same as the numerical value of the triangular area of the region enclosed by the X-axis, the line segment X'Q' and the line segment OQ'.]

b. When $x = 5$ find the position of T so that P' T is a line tangent to the curve A at P'.

c. Find the slope of the line tangent to the curve A at P' when $x=5$.

d. Find the slope of the line tangent to the curve A at P' when $x = 1$, 2, and 4.

e. Find the slope of the line tangent to the curve A at P' when $x = a$.

3. Suppose the right triangle OXQ as given in Figure 6 has side OX of length 8 on the horizontal axis and vertical side XQ of length 8. Above triangle OXQ sketch the area curve A, which is related to the line Y determined by vertices O and Q.

a. Let *x, q* and *p* be defined as follows:
- *x* denotes the length of segment OX',
- *q* denotes the length of segment X'Q', and
- *p* denoted the length of segment X'P'= *p*

Give an equation relating *q* and *x*. Give an equation relating *p* and *x*. [Remember the numerical value of the length of X'P' is the same as the numerical value of the triangular area of the region enclosed by the X-axis, the line segment X'Q' and the line segment OQ'.]

b. When $x = 5$ find the position of T so that P' T is a line tangent to the curve A at P'.

c. Find the slope of the line tangent to the curve A at P' when x=5.

d. Find the slope of the line tangent to the curve A at P' when $x = 1$, 2, 3, and 4.

e. Find the slope of the line tangent to the curve A at P' when $x = a$.

4. Suppose the right triangle OXQ as given in Figure 6 has side OX of length L (>1) on the horizontal axis and vertical side XQ of length M. Above triangle OXQ sketch the area curve A, which is related to the line Y determined by vertices O and Q.

a. Let x, q and p be defined as follows:

- x denotes the length of segment OX',
- q denotes the length of segment X'Q', and
- p denoted the length of segment X'P'= p

Give an equation relating q and x. Give an equation relating p and x. [Remember the numerical value of the length of X'P' is the same as the numerical value of the triangular area of the region enclosed by the X-axis, the line segment X'Q' and the line segment OQ'.]

b. When $x = 1$ find the position of T so that P' T is a line tangent to the curve A at P'.

c. Find the slope of the line tangent to the curve A at P' when $x = a$.

5. Show that the line tangent to the graph of $Y= CX^2$ at the point (a,Ca^2) where $a>0$ has slope $2Ca$. [Hint: Relate the curve to a right triangle with legs of length L and 2CL and use Barrow's Theorem.]

Other exercises to cover some of the following trails with minimum or no reliance on functions:

Connection between slope and rates and velocity.

rates for downloading software, files and mp3's as function of time : size of file downloaded at time t: internet connection speed.

connection be **velocity and accumulated distance**

coonection between rates and accumulated trash

max/ mins and graphing ??

For areas and arc length - simple polygons and estimations of curved figures Maybe based on triangles - Connect to archimedes and area of circle. Maybe Euclid. Area of parabola for estimation. or maybe by Don Albers Aproach.

Probability with a constant density. Uniform distribution. Step distribution.

[Add more exercises/investigations related to rates and accumulations- maybe connecting with the graphical.]

0.A APPENDIX: THE GEOMETRIC PROOF OF BARROW'S THEOREM

We begin by recalling the statement of Barrow's Theorem.

Theorem:

Hypotheses:

Suppose Y is a curve intersecting the line OX only at the point O.

Suppose further that the curve A has the following properties:

- For a point P chosen on the curve A there is a corresponding point Q on the curve Y so that the segment PQ intersects the line OX at the point R forming a right angle.
- The numerical value of the length of segment PR, p, is the same as the numerical value of the area of the region enclosed by the curve Y, the segment OR and segment RQ, q, so $p = q$. (See Figure 7.)

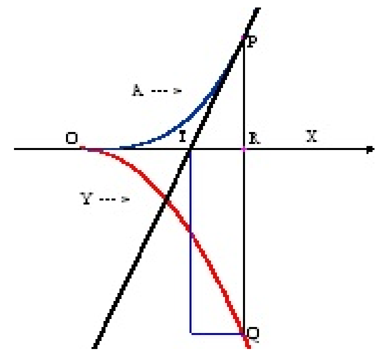

Figure 7

Suppose the point T is chosen on the segment OR so that the numerical value of the area of the rectangle with sides TR and RQ, which we'll denote t, is equal to p.

Conclusion: The line TP is tangent to the curve A at the point P.

Legend for variables used in proof of Barrow's Theorem

p=length of segment PR	r=length of segment UR	w=length of segment TR
p*=length of segment P*R*	s=length of segment SU	x=length of segment OR
q=area betw y and ORQ	t=area or rect det by T and Q	x*=length of segment OR*
q*=area betw y and OR*Q*	u=length of segment PU	y=length of segment RQ
	u*=length of segment P*U	

Proof: We follow the ideas in Barrow's original presentation. Beware that his proof is geometric, and we are presenting some of it algebraically.

To show that TP is tangent to the curve A at the point P, we'll show that Pis the only point on both the curve A and the line TP. We'll do this by considering a point P* ($\neq$ P)on the curve A and show that the line TP doesn't pass through P*.

For convenience [like 7-11], let's assume that if the point P is moved to the right, on the curve A, then the lengths of the segments OR, x, and RQ, y, increase. Furthermore, we'll assume that P* is some point on A between O and P.

Draw a line parallel to the X axis through P*. Now that you have constructed this line, label the point where this line intersects TP as S, and the point where it intersects PR as U. (See Figure 8.)

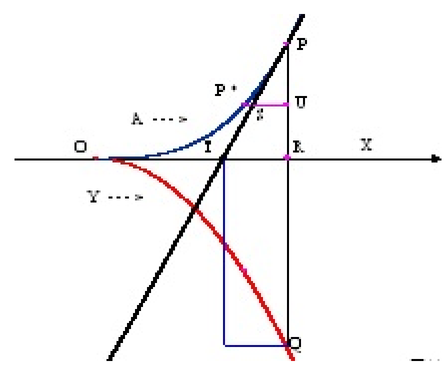

Figure 8

Recall that T was selected so that $t = p$, and so $p = y * w$ where w is the numerical value of the length of segment TR. Thus $y = p/w$.

Notice that triangle TPR is similar to triangle SPU, and since corresponding sides of similar triangles are proportional we have

$p/w = u/s$ where u is the numerical value of the length of segment PU and s is the numerical value of the length of segment SU.

Combining these last two equalities using the transitive property, we now have $y = u/s$.

Multiplying both sides of this equation by s we have that $y \bullet s = u$.

Draw a line through P* parallel to line PQ labeling the point where this line intersects line OR as R*, and the point where this line intersects the curve Y as Q*. Because of the defining properties of the curve A, since P* is on the curve A and R* is on OX, and Q* is on curve Y, then the numerical value of the length of P*R*, p^*, is equal to the numerical value of the area of the region enclosed by the curve Y, segment OR*, and segment R*Q*, q^*. (See Figure 9.)

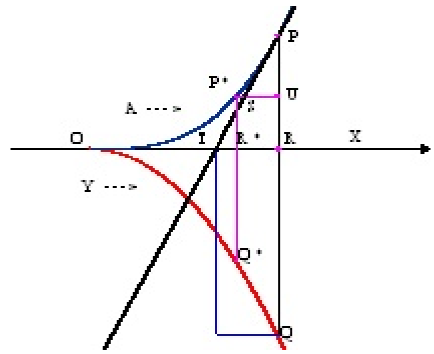

Figure 9

Now because quadrilateral P*R*RU is a rectangle, then $p^* = r$ where r is the numerical value of the length of segment UR. Since $p = u + r$, then $u = p - r = p - p^*$. Subtracting corresponding areas with equal numerical value, we find that u must equal the numerical value of the region enclosed by line segments R*Q*, R*R, RQ, and the curve Y between Q and Q*. [See Figure 10.]

But this region is contained in a rectangle determined by R*R and RQ. [What assumption justifies this?] So u is less than the product $y(x\text{-}x^*)$ where x^* is the numerical value of the length of segment OR*. Recall that $u = y \bullet s$.

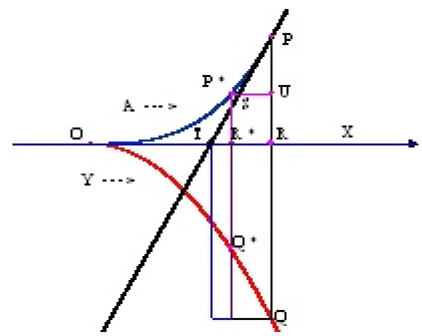

Figure 10

Thus we can conclude that $y \bullet s. < y.\bullet(x\text{-}x^*)$ and therefore $s < (x\text{-}x^*)$. But $(x\text{-}x^*) = u^*$ where u^* is the numerical value of the length of segment P*U , so $s < u^*$, *i.e.*, segment SU is shorter then segment P*U, showing that P* ≠ S.

A similar argument (left as an exercise) shows that if P* is to the right of P then P* is also not on the line TP.

EOP.