Mapping Figures Workshop
University of Utah
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Session I Linear Mapping Figures

We begin our introduction to mapping figures by a consideration of linear functions :

“ $y = f(x) = mx + b$ ”

Mapping Figures

A.k.a.
Function Diagrams
Dynagraphs

Written by Howard Swann and John Johnson

A early source for visualizing functions at an elementary level before calculus.

This is copyrighted material!

YOUR FRIENDLY NEIGHBORHOOD FUNCTION CONSISTS OF TWO SETS AND A BUNCH OF ARROWS THAT OBEY

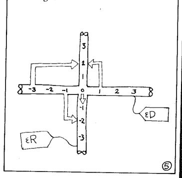
RULE **1** THE ARROWS ALWAYS START FROM THE SAME SET, CALLED THE **DOMAIN** AND GO TO THE OTHER SET, CALLED THE **RANGE**.

RULE **2** EVERYTHING IN THE DOMAIN-SET MUST HAVE EXACTLY ONE ARROW FROM IT. EVERYTHING IN THE RANGE-SET MUST HAVE AT LEAST ONE ARROW TO IT.
(IT'S OK, TO HAVE 2 ARROWS TO 1 THING.)

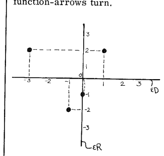
So two or more arrows can hit the same thing in the range-set, but only one arrow can come from any particular thing in the domain-set.

Using arrows in the RULES unfortunately has its drawbacks — as functions become more elaborate, the arrows can get pretty difficult to follow . . . .

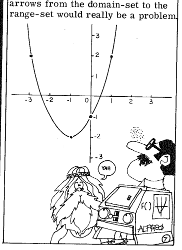
Fix up each arrow so it starts off straight up or down and then makes a right-angle turn directly over to the range.



Now we can preserve ALL the information about the function by just keeping the DOTS where the function-arrows turn.



So the dots tell us all about the function. There are usually many more arrows than we have shown and thus more dots. In fact, usually there are so many that the dots make a solid curve. Showing how such a function works using just arrows from the domain-set to the range-set would really be a problem.

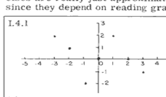


Remember that the DOMAIN of any function is always part of the horizontal ⟷ line, called the "x-axis" because any arbitrarily chosen thing in the domain-set is usually called "x." The RANGE is always part of the vertical (↕) line, called the "y - axis" because an arbitrary thing in the range-set is usually called "y."

EXERCISES

The answers to most of these exercises are really just approximate, since they depend on reading graphs.

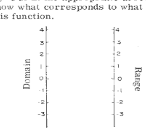
I.4.1



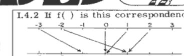
(a) Label the dots.
(b) If the dots represent a function, what is

f(3) = ?
f(-3) = ?
f(1) = ?
f(2) = ?
f(0) = ?

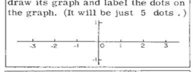
(c) Put in the appropriate arrows to show what corresponds to what in this function.



I.4.2 If f( ) is this correspondence;

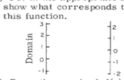


draw its graph and label the dots on the graph. (It will be just 5 dots.)

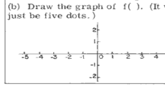


I.4.3 If the domain of f( ) is $\{-1, 0, \frac{1}{2}, 2, 3\}$ and $f(-1) = -1$, $f(0) = 0$, $f(\frac{1}{2}) = 1$, $f(2) = -2$ and $f(3) = -2$,

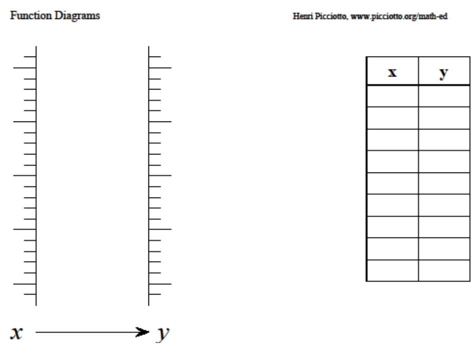
(a) Put in the appropriate arrows to show what corresponds to what in this function.

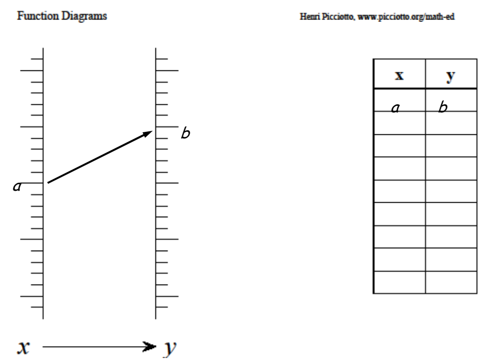


(b) Draw the graph of f( ). (It will just be five dots.)



Function Diagrams by Henri Picciotto





Mapping Diagrams and Functions

- SparkNotes › Math Study Guides › Algebra II: Functions Traditional treatment.
  - http://www.sparknotes.com/math/algebra2/functions/
- **Function Diagrams. by Henri Picciotto Excellent Resources!**
  - **Henri Picciotto's Math Education Page**
  - **Some rights reserved**
- Flashman, Yanosko, Kim https://www.math.duke.edu//education/prep02/teams/prep-12/

Outline of Remainder of Morning…

- Linear Functions: They are everywhere!
- Tables
- Graphs
- Mapping Figures
- Excel, Winplot and other technology Examples
- Characteristics and Questions
- Understanding Linear Functions Visually.

Visualizing Linear Functions

- Linear functions are both necessary, and understandable- even without considering their graphs.
- There is a sensible way to visualize them using "mapping figures."
- Examples of important function features (like "slope" and intercepts) will be illustrated with mapping figures.
- Examples of activities for students that engage understanding both function and linearity concepts.
- Examples of these mappings using simple straight edges as well as technology such as Winplot (freeware from Peanut Software), Geogebra, and possibly Mathematica and GSP
- Winplot is available from http://math.exeter.edu/rparris/peanut/

Linear Functions: They are everywhere!

- Where do you find Linear Functions?
  - At home:
  - On the road:
  - At the store:
  - In Sports/ Games

Linear Functions: Tables

x	5 x - 7
3	
2	
1	
0	
-1	
-2	
-3	

Complete the table.
x = 3,2,1,0,-1,-2,-3
f(x) = 5x – 7

f(0) = ___?

For which x is f(x) > 0?

Linear Functions: Tables

X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

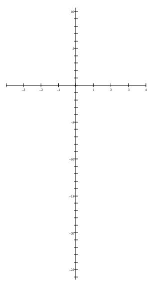
Complete the table.
x = 3,2,1,0,-1,-2,-3
f(x) = 5x – 7

f(0) = ___?

For which x is f(x) > 0?

Linear Functions: On Graph

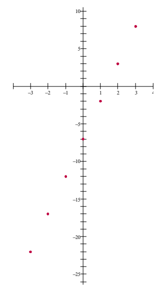
Plot Points (x , 5x - 7):



X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Linear Functions: On Graph

Connect Points (x , 5x - 7):



X	5 x - 7
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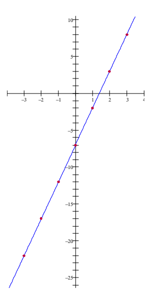
Linear Functions: On Graph



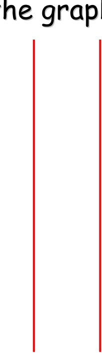
Connect the Points

X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Linear Functions: Mapping Figures
What happens before the graph.

- Connect point x to point 5x – 7 on axes

X	5 x – 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22



Linear Functions: Mapping Figures
What happens before the graph.

X	5 x - 7
3	8
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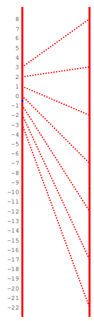


Function-Equation Questions
with mapping figures

- Solving a linear equations: 2x+1 = 5

  2x+1 = x + 2
  - f(x) = 2x+1: For which x does f(x) = 5 ?
  - g(x) = x+2: For which x does f(x) = g(x)?
- Find "fixed points" of f : f(x) = 2x+1
  - For which x does f(x) = x?

Simple Examples are important!

- f(x) = x + C Added value: C
- f(x) = mx Scalar Multiple: m

  Interpretations of m:
  - slope
  - rate
  - Magnification factor
  - m > 0 : Increasing function
  - m = 0 : Constant function [WS Example]
  - m < 0 : Decreasing function [WS Example]

Simple Examples are important!

f(x) = mx + b with a mapping figure -- Five examples:

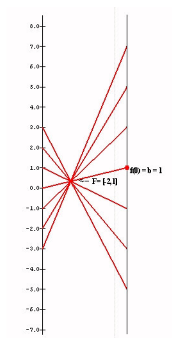
- Example 1: m =-2; b = 1: f(x) = -2x + 1
- Example 2: m = 2; b = 1: f(x) = 2x + 1
- Example 3: m = $\frac{1}{2}$; b = 1: f(x) = $\frac{1}{2}$ x + 1
- Example 4: m = 0; b = 1: f(x) = 0 x + 1
- Example 5: m = 1; b = 1: f(x) = x + 1

Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

Example 1: m = -2; b = 1
f (x) = **-2x + 1**

- Each arrow passes through a single point, which is labeled **F = [- 2,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x,* on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, -2*x* + 1,

    which corresponds to the linear function's value for the point/number, *x*.



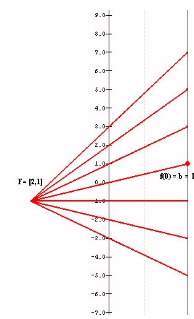
Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

Example 2: *m* = 2; *b* = 1
f(x) = 2x + 1

Each arrow passes through a single point, which is labeled
F = [2,1].

- The point **F** completely determines the function *f*.
  - **given a point** / number, *x,* on the source line,
  - there is a **unique arrow passing through F**
  - **meeting** the target line at **a unique point** / number, 2*x* + 1,

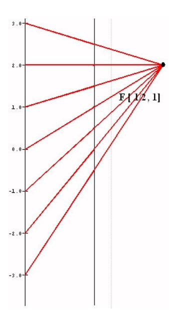
  which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

- **Example 3: *m* = 1/2; *b* = 1**
  f(x) = ½ x + 1
- Each arrow passes through a single point, which is labeled **F = [1/2,1].**
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x,* on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, ½ *x* + 1,

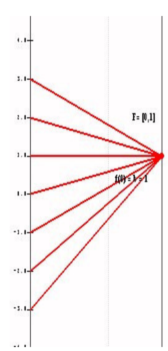
    which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples:

- **Example 4: *m* = 0; *b* = 1**
  f(x) = 0 x + 1
- Each arrow passes through a single point, which is labeled **F = [0,1].**
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x,* on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, *f*(*x*)=1,

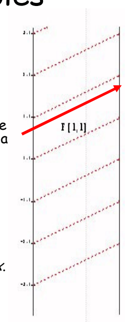
    which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = mx + b$ with a mapping figure -- Five examples

Example 5: $m = 1$; $b = 1$

$f(x) = x + 1$



- Unlike the previous examples, in this case it is not a single point that determines the mapping figure, but the single arrow from 0 to 1, which we designate as F[1,1]
- It can also be shown that this single arrow completely determines the function. Thus, given a point / number, x, on the source line, there is a unique arrow passing through ***x parallel to*** F[1,1] meeting the target line a unique point / number, $x + 1$, which corresponds to the linear function's value for the point/number, x.
  - The single arrow completely determines the function f.
    - **given a point** / number, x, on the source line,
    - there is a **unique arrow** through ***x parallel to*** F[1,1]
    - **meeting** the target line at **a unique point** / number, $x + 1$,

which corresponds to the linear function's value for the point/number, x.

End of Session I

- Questions
- Break - food and thought
- Partner/group integration task

Function-Equation Questions

with linear focus points

- Solve a linear equations:
  - 2x+1 = 5
  - 2x+1 = -x + 2
  - Use focus to find x.
- "fixed points" : f(x) = x
  - Use focus to find x.

Morning and Lunch Break: Think about These Problems (in Groups 1-2; 3-4)

M.1 How would you use the Linear Focus to **find the mapping figure for the function inverse for a linear function when m≠0?**

M.2 How does the **choice of axis scales** affect the **position of the linear function focus point** and its use in solving equations?

M.3 Describe the visual features of the mapping figure for the quadratic function $f(x) = x^2$.
How does this generalize for *even* functions where **$f(-x) = f(x)$?**

M.4 Describe the visual features of the mapping figure for the cubic function $f(x) = x^3$.
How does this generalize for *odd* functions where **$f(-x) = -f(x)$?**

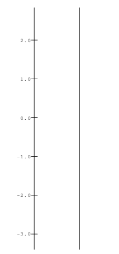
Session II More on Linear Mapping Figures

We continue our introduction to mapping figures by a consideration of the **composition of linear functions**.

Compositions are keys!

An example of composition with mapping figures of simpler (linear) functions.

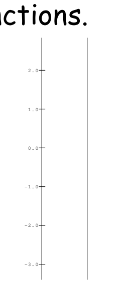
- $g(x) = 2x$; $h(u)=u+1$
- $f(x) = h(g(x)) = h(u)$ where $u = g(x) = 2x$
- $f(x) = (2x) + 1 = 2x + 1$
  - $f(0) = 1$ slope = 2



Compositions are keys!

Linear Functions can be understood and visualized as compositions with mapping figures of simpler linear functions.

- $f(x) = 2x + 1 = (2x) + 1$:
  - $g(x) = 2x$; $h(u)=u+1$
  - $f(0) = 1$ slope = 2



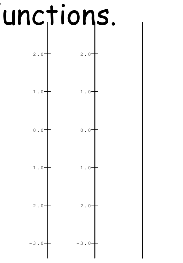
Compositions are keys!

Linear Functions can be understood and visualized as compositions with mapping figures of simpler linear functions.

Example: $f(x) = 2(x-1) + 3$

$g(x)=x-1$ $h(u)=2u$; $k(t)=t+3$

- $f(1)= 3$ slope = 2



Inverses, Equations and Mapping Figures

- Inverse: If f(x) = y then invf(y)=x.
- So to find invf(b) we need to find any and all x that solve the equation f(x) = b.
- How is this visualized on a mapping figure?
- Find b on the target axis, then trace back on any and all arrows that "hit"b.

Mapping Figures and Inverses

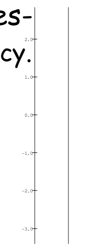
Inverse linear functions:

- Use transparency for mapping figures-
  - Copy mapping figure of g to transparency.
  - Flip the transparency to see mapping

  figure of inverse function g.

  ("before or after")

  invg(g(a)) = a; g(invg(b)) = b;
- Example i: g(x) = 2x; invg(x) = 1/2 x
- Example ii: h(x) = x + 1 ; invh(x) = x - 1

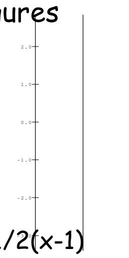


Mapping Figures and Inverses

Inverse linear functions:

- socks and shoes with mapping figures
- g(x) = 2x; invf(x) = 1/2 x
- h(x) = x + 1 ; invh(x) = x - 1

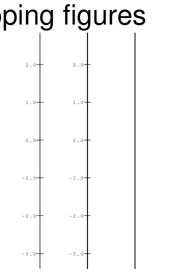
- f(x) = 2 x + 1 = (2x) + 1 =h(g(x))
  - g(x) = 2x; h(u)=u+1
  - inverse of f: invf(x)=invg(invh(x))=1/2(x-1)



Mapping Figures and Inverses

Inverse linear functions:

- "socks and shoes" with mapping figures
- f(x) = 2(x-1) + 3:
  - g(x)=x-1 h(u)=2u; k(t)=t+3
  - Inverse of f: 1/2(x-3) +1



End of Session II

- Questions
- Lunch Break - food and thought
- Partner/group integration task

Lunch Break: Think about These Problems (in Groups 1-3; 4-5)

L.1 **Describe the visual features of the mapping figure for the quadratic function** $f(x) = x^2$.
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? "Infinity"?

L.2 Describe the visual features of the mapping figure for the quadratic function $f(x) = A(x-h)^2 + k$ using composition with simple linear functions.
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? "Infinity"?

L.3 **Describe the visual features of a mapping figure for the square root function** $g(x) = \sqrt{x}$ **and relate them to those of the quadratic** $f(x) = x^2$.
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? "Infinity"?

L.4 **Describe the visual features of the mapping figure for the reciprocal function** $f(x) = 1/x$.
Domain? Range? "Asymptotes" and "infinity"? Function Inverse?

L.5 Describe the visual features of the mapping figure for the linear fractional function $f(x) = A/(x-h) + k$ using composition with simple linear functions.
Domain? Range? "Asymptotes" and "infinity"? Function Inverse?

Session III More on Mapping Figures: Quadratic, Exponential and Logarithmic Functions

We continue our introduction to mapping figures by a consideration of **quadratic, exponential and logarithmic functions.**

Examples on Excel, Winplot, Geogebra

- Excel example:
- Winplot examples:
  - Linear Mapping examples
- Geogebra examples:
  - dynagraphs.ggb
  - Composition

Web links

- https://www.math.duke.edu//education/prep02/teams/prep-12/
- http://users.humboldt.edu/flashman/TFLINX.HTM
- http://www.dynamicgeometry.com/JavaSketchpad/Gallery/Trigonometry_and_Analytic_Geometry/Dynagraphs.html
- http://demonstrations.wolfram.com/Dynagraphs/
- http://demonstrations.wolfram.com/ComposingFunctionsUsingDynagraphs/

Quadratic Functions

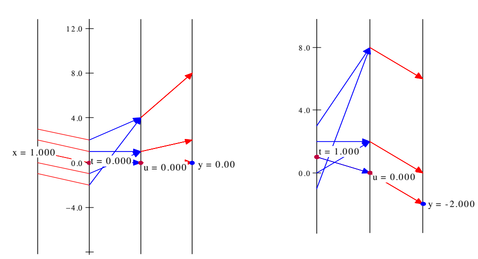
- Usually considered as a key example of the power of analytic geometry- the merger of algebra with geometry.
- The algebra of this study focuses on two distinct representations of of these functions which mapping figures can visualize effectively to illuminate key features.
  - $f(x) = Ax^2 + Bx + C$
  - $f(x) = A(x-h)^2 + k$

Examples

- Use compositions to visualize
  - $f(x) = 2(x-1)^2 = 2x^2 - 4x + 2$
  - $g(x) = 2(x-1)^2 + 3 = 2x^2 - 4x + 5$
- **Observe how even symmetry is transformed.**
- **These examples illustrate how a mapping figure visualization of composition with linear functions can assist in understanding other functions.**

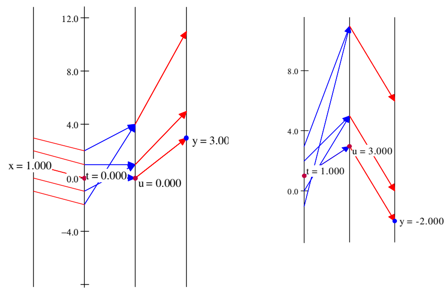
Quadratic Mapping Figures

$f(x) = 2(x-1)^2 \qquad = 2x^2 - 4x + 2$



Quadratic Mapping Figures

$g(x) = 2\,(x\text{-}1)^2 + 3 \qquad = 2x^2 \text{-}4x + 5$



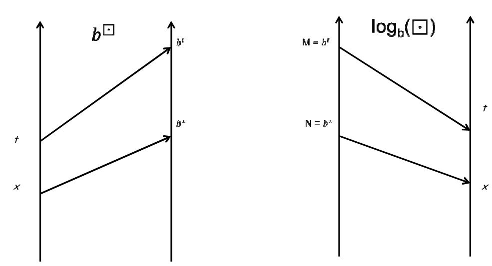
Quadratic Equations and Mapping Figures

- To solve $f(x) = Ax^2 + Bx + C = 0$.
- Find 0 on the target axis, then trace back on any and all arrows that "hit" 0.
- Notice how this connects to x = -B/(2A) for symmetry and the issue of the number of solutions.

Definition

- Algebra Definition
  $b^L = N$ if and only if $\log_b(N) = L$
- Functions:
- $f(x)= b^x = y$; $\text{invf}(y) = \log_b(y) = x$
- $\log_b = \text{invf}$

Mapping figures for exponential functions and "inverse"



Visualize Applications with Mapping Figures

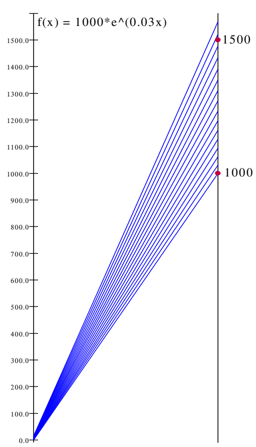
"Simple" Applications

I invest $1000 @ 3% compounded continuously. How long must I wait till my investment has a value of $1500?

Solution: $A(t) = 1000\ e^{0.03t}$.

Find t where $A(t) = 1500$.

Visualize this with a mapping figure before further algebra.



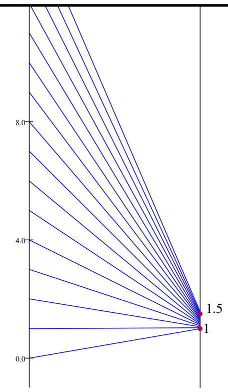
"Simple" Applications

Solution: $A(t) = 1000\ e^{0.03t}$.

Find t where $A(t) = 1500$.

Algebra: Find t where $u=0.03t$ and $1.5 = e^u$.

Consider simpler mapping figure on next slide



"Simple" Applications

Solution: $A(t) = 1000\ e^{0.03t}$.

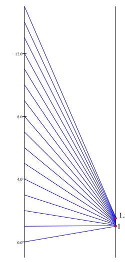
Find t where A(t) = 1500.

Algebra: Find t where $u=0.03t$ and $1.5 = e^u$.

Consider simpler mapping figure and solve with logarithm:

$u=0.03t = \ln(1.5)$ and

$t = \ln(1.5)/0.03 \approx 13.52$



Example: Using Mapping Figures in "Proof" for Properties of Logs.

$e^{t+x} = e^t e^x = u*y$ where $u = e^t$ and $y= e^x$.

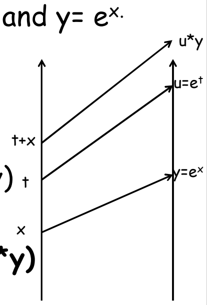
Thus by definition:

$x = \ln y$; $t = \ln u$;

And $t+x = \ln(u*y)$

SO

$\ln u + \ln y = \ln(u*y)$



End of Session III

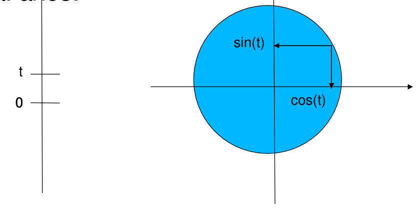
- Questions
- Break - food and thought
- Partner/group integration task

Session IV More on Mapping Figures: Trigonometry and Calculus Connections

We complete our introduction to mapping figures by a consideration of **trigonometric functions and some connections to calculus.**

Seeing the functions on the unit circle with mapping figure.

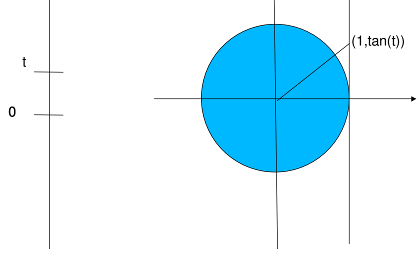
Sine and cosine of t measured on the vertical and horizontal axes.



Note the visualization of periodicity.

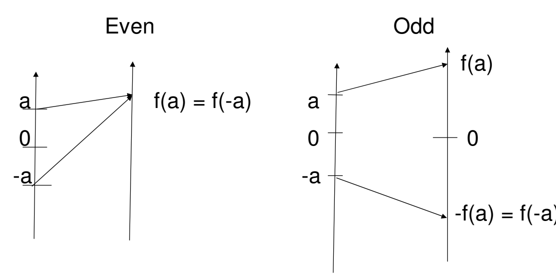
Tangent Interpreted on Unit Circle

- Tan(t) measured on the axis tangent to the unit circle.



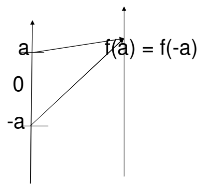
- Note the visualization of periodicity.

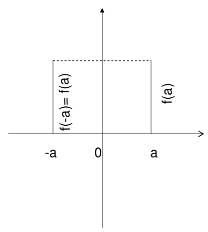
Even and odd on Mapping Figures



Even Function Mapping Figures and Graphs

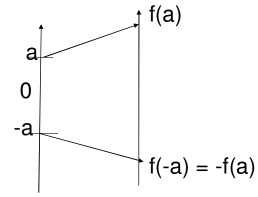
An Even Function

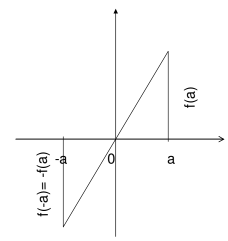




Odd Function Mapping Figures and Graphs

An Odd Function



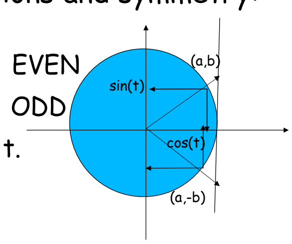


Trigonometric functions and symmetry:

cos(-t) = cos(t) for all t. EVEN

sin(-t) = -sin(t) for all t. ODD

tan(-t) = - tan(t) for all t.



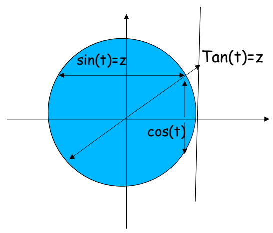
Justifications from unit circle mapping figures for sine, cosine and tangent..

Trig Equations and Mapping Figures

- To solve ***trig(x)*** = ***z***.
- Find z on the target axis, then trace back on any and all arrows that "hit" z.
- Notice how this connects to periodic behavior of the trig functions and the issue of the number of solutions in an interval.
- This also connects to understanding the inverse trig functions.

Solving Simple Trig Equations:

Solve trig(t)=z from unit circle mapping figures for sine, cosine and tangent.



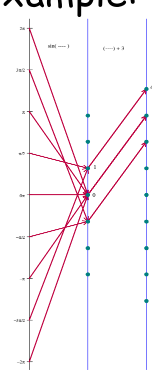
Winplot Examples for

[Trig Functions]

[Trig Linear Compositions]

Compositions with Trig Functions

Example: y = f(x) =sin(x) + 3



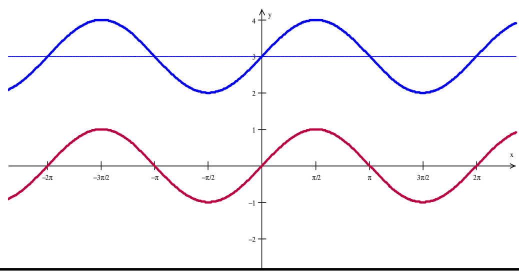
- Mapping figure for
- y =f(x) = sin(x) + 3 considered as a composition:
- First: u = sin(x)
- Second: y =u+3 so the result is
- y = (sin(x)) + 3

Example: Graph of y = sin(x) + 3

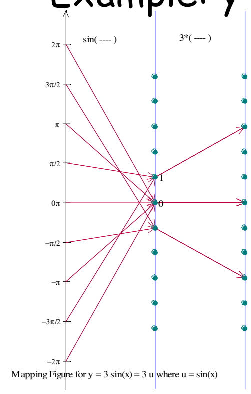
Graph of y = sin(x) + 3 [Winplot]

Amplitude: 1

Period: 2π



Example: y = f(x) = 3 sin(x)



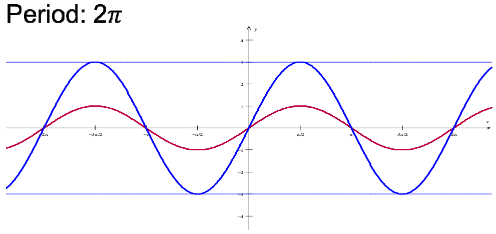
- Mapping figure for
- y =f(x) = 3 sin(x) considered as a composition:
- First: u = sin(x)
- Second: y = 3u so the result is
- y = 3 (sin(x))

Example: Graph of y = 3 sin(x)

Graph of y = 3 sin(x) [Winplot]

Amplitude: 3

Period: 2π



Interpretations

- y= 3sin(x):

t -> (cos(t), sin(t)) -> (3cos(t),3sin(t))

unit circle magnified to circle of radius 3.

- Y = sin(x) + 3:

t -> (cos(t), sin(t)) -> (cos(t), sin(t)+3)

unit circle shifted up to unit circle with center (0,3).

Show with winplot: circles_sines.wp2;

Scale change before trig.

Mapping figures and graphs for $f(x) = \sin(Bx)$

- Amplitude and period

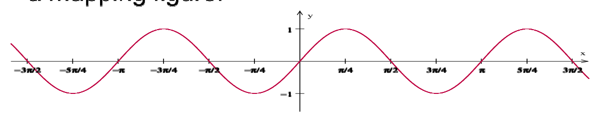
Connection to solving equations:

- Example: $\sin(2x) = 1$;
  - $2x = \pi/2, 5\pi/2$
  - $x = \pi/4$, $5\pi/4$
  - Difference is period: $(5\pi - \pi)/4 = \pi$.

Scale change before trig.

Mapping figures and graphs for $f(x) = \sin(2x)$

Amplitude:1 Period: π
Use Excel here to demonstrate composition and a mapping figure.



Interpretations of these functions with circles.

Show with winplot: dot_races.wp2
on Moodle:Dot races! (winplot)

Period for $Y = \sin(Bx)$: $2\pi/B$

Scale change before trig

Mapping figures and graphs for $f(x) = \sin(x+ D)$ or

- Amplitude and period and shift.

Connection to solving equations:

- Example: $\sin(x +\pi/3) = 0$;
  - $x+\pi/3 = 0$
  - $x = -\pi/3$

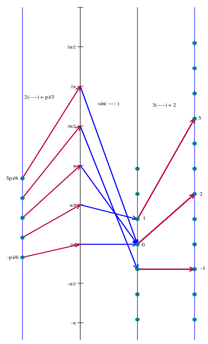
Shift of sine curve to start at $x = -\pi/3$: $(-\pi/3,0)$

- Interpretations of these functions with circles.

Altogether!

- $f(x) = 3 \sin(2x+\pi/3) + 2$
- Mapping figure: Before $u = 2x+\pi/3$
- After $y = 3z + 2$
- MIDDLE: $z = \sin(u)$.
- Amplitude :3, period: π , and shift: ???.
- Visualize on circle. Dot races and mapping figures.
- Solve equations for period and shift.
- $u = 0$ and $u = 2\pi$. Period = difference in x.

Mapping figure



$f(x) = 3 \sin(2x+\pi/3) + 2$

Mapping figure:

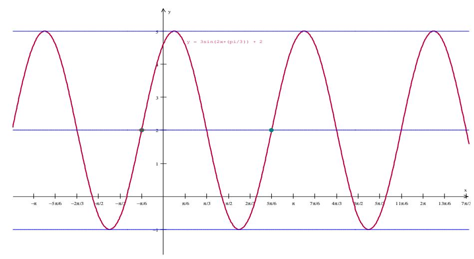
Before $u = 2x+\pi/3$

After $y = 3z + 2$

MIDDLE: $z = \sin(u)$.

Graph

- $f(x) = 3 \sin(2x+\pi/3) + 2$



Thanks

The End!

☺

Questions?

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More References

More References

- Goldenberg, Paul, Philip Lewis, and James O'Keefe. "Dynamic Representation and the Development of a Process Understanding of Function." In The Concept of Function: Aspects of Epistemology and Pedagogy, edited by Ed Dubinsky and Guershon Harel, pp. 235–60. MAA Notes no. 25. Washington, D.C.: Mathematical Association of America, 1992.

More References

- http://www.geogebra.org/forum/viewtopic.php?f=2&t=22592&sd=d&start=15
- "Dynagraphs}--helping students visualize function dependency • GeoGebra User Forum
- "degenerated" dynagraph game ("x" and "y" axes are superimposed) in GeoGebra: http://www.uff.br/cdme/c1d/c1d-html/c1d-en.html

Thanks
The End!   REALLY!

☺

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