MAA Session on Bridging the Gap: Designing an Introduction to Proofs Course

Understanding the Problem: Unification, Generalization or Abstraction?

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Another Presentation

- **Saturday January 12, 2013, 9:00 a.m.**
  MAA Session on Fostering Mathematical Habits of Mind, II
  Room 6E, Upper Level, San Diego Convention Center

***"The Benefits of A Habit:
Examining Evidence to Understand Statements and Proofs."***

Acknowledgement

This work is based in part on

recent experiences teaching at HSU: Math 240 (3 units) **Introduction to Mathematical Thought and Math 381 (1 unit) Tutorial in Writing Proofs**

Using

Daniel Solow's **How To Read and Do Proofs (Solow 1) and**

The Keys to Advanced Mathematics... (Solow 2)

and ***How To Solve It*** by G. Polya.

Initial Thoughts
The Big Picture

- Polya’s 4 Phases of Problem Solving

“How To Solve It" by G. Polya**.**

1. Understand the problem.
2. See connections to devise a plan.
3. Carry out the plan.
4. Look back. Reflect on the process and results.

Organizing A Course on Proofs I

- What I don't emphasize:
- A lengthy discussion of logic and truth tables. [This is not a course in Logic.]
- Venn Diagrams and proving set equalities.

[This is not a course in Set Theory.]

Organizing A Course on Proofs II

- Parts of mathematics I do cover as illustrative of Mathematical Thinking:
- Arithmetic. Primes, division, and factors. [This is not a course in Number Theory.]
- Rational, real and complex numbers. Operations, order, open sets.

[This is not a course in calculus, analysis, or topology]

- Finite and infinite sets. [This is not a course in set theory.]
- Functions: Discrete and continuous. [This is not a discrete mathematics course.]
- Counting: finite and infinite. [This is not a course in combinatorics.]

One Approach to Proof Courses

Focus on organization of proofs (Solow and others)

- Brief look at “logic”
- Look for “key questions.”
- Conditional statements
  - “Assume… show…”
- Universal statements
  - “Choose … show…”
- Existential statements
  - “Construct … show…”
- Indirect Arguments.
  - Contrapositive
  - Contradiction
- Special techniques:
  - Induction (Use of natural number and order.)
  - Uniqueness
  - Alternatives

Another View…

1. Understand the problem.
2. See connections to devise a plan.

"Generalize!"

What do we mean when we say,

"**Generalize** this example, definition, argument, …"

Polya on generalization

- “ … passing from the consideration of one object to the consideration of a set containing that object;
- or passing from the consideration of a restricted set to that of a more comprehensive set containing the restricted one.”

“Generalization” in Problem Solving.

1. Generalize to understand.
2. Generalize to devise a plan.
3. Generalize to execute a plan.
4. Generalize in reflection.

Refining the concept of “generalization”

Daniel Solow describes three “uses of mathematics”:

1. Unification
2. Generalization
3. Abstraction

Unification [Solow]

Unification: “The process of creating …[a] new, encompassing representation… . This technique involves combining two or more concepts (problems, theories, and so on) into a single framework from which you can study each of the special cases.”

Generalization [Solow]

"**Generalization** is the process of creating, from an original concept (problem, definition, theorem, and so on), a more general concept (problem, definition, theorem, and so on) that includes not only the original one, but many other new ones as well."

Abstraction [Solow]

“**Abstraction** is the process of taking the focus farther and farther away from specific items by working with general *objects*.”

Context

It is important to understand the role that **context** plays with these three uses in mathematical thought.

Comment: Context was a key element of Bertrand Russell’s analysis of the use of language in “On Denoting”.
It is an important element in understanding interpretations in foundations and model theory.

Examples of Context (U, G, A)

- The Natural Numbers U
- The Integers U G
- The Rational Numbers U G
- The Real Numbers U G
- The Complex Numbers U G
- The Euclidean Plane U
- The Cartesian Plane U
- The Cartesian (Euclidean) “n – space” G

More Contexts

- Sequences of (Rational) (Real) (Complex) Numbers U, G
- Polynomials with *** Coefficients U, G
- Real (Complex) -valued Functions of Real (Complex) Numbers U, G
- Permutations of {1,2,3,…, n} G
- Planar (Conic) Curves. U, G
- Real-valued Multivariable Functions of Real Numbers. G
- Linear Operators on Euclidean n-Space G

Even more Contexts

- A Set (Universe). Functions on Sets. A
- A Group, Ring, Field. “Homomorphisms” A
- A Vector Space. “Linear Transformations” A
- An Inner Product Space. A
- A Metric Space. “Isometries” A
- A Topological Space. “Continuous functions” A
- A Measure Space. “Measurable functions” A
- A Category. Objects, Maps. A

Context Connections

- **U**nification: A single context.
- **G**eneralization: Distinct but connected, related contexts.
- **A**bstraction: Broad context definition with a structural characterization that allows discourse with limited specificity.

Changing Approach to Proof

Changed Approach to Proof Course

- Balanced Focus
  - **Polya steps**
  - **Context and organization**

1. **<u>Understand the context</u>** of the statement(s).
   - Is this a **unification,**
   - **generalization, or**
   - **abstraction?**
2. **R**ecognize how to connect understanding of instances and the context to **develop a plan** with an appropriate **<u>organization</u>**.
3. **Execute the plan** at the appropriate level of discourse and with a clear **<u>organization</u>.**

4. Reflect on the role of context and how that effected the plan and its execution.

Compare Approaches

Previous Approach to Proof Course

- Primary focus on organization of proofs (Solow)
  - Brief look at “logic”
  - Look for “key questions.”
  - Conditional statements
    - “Assume… show…”
  - Universal statements
    - “Choose … show…”
  - Existential statements
    - “Construct … show…”
  - Indirect Arguments.
    - Contrapositive
    - Contradiction
  - Special techniques:
    - Induction (Use of natural number and order.)
    - Uniqueness
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Examples of Unification

Addition of vectors: Define (a,c) +(b,d) to be the vector pair (a+b, c+d).
Fact: (3,4)+(0,0) = (3,4)
Fact: (-4,7)+(0,0) = (-4,7)
Fact: (8,-5)+(0,0) = (8,-5)
Unification: For any a and c, (a,c)+(0,0) = (a,c)

Fact: (3,4)+(-3,-4) = (0,0)
Fact: (-4,7)+(4,-7) = (0,0)
Fact: (8,-5)+(-8,5) = (0,0)
Unification: For any (a,b), (a,b)+(-a,-b) = (0,0)

Inner Product of Vectors: Define (a,c) ∘ (b,d) to be the number a·b + c·d.
Fact: (3,4)∘(-4,3) = 0
Fact: (-4,7)∘(-7,-4) = 0
Fact: (8,-5)∘(5,8) = 0
Unification: For any (a,b), (a,b) ∘ (-b,a) = 0

Examples of Generalization

Generalization of Addition:
Define (a1,a2, …, an) +(b1,b2,…, bn) to be the n-tuple (a1+b1, a2+b2, …, an+bn).

Unification: For any a and b, (a,b)+(0,0) = (a,b)
Generalization: For any a1,a2, …, an, (a1,a2, …, an) +(0,0, …, 0) = (a1,a2, …, an)

Unification: For any (a,b), (a,b)+(-a,-b) = (0,0)
Generalization: For any (a1,a2, …, an) ,
(a1,a2, …, an) + (-a1,-a2, …, -an) = (0,0, … , 0)

Generalization of Inner Product of Vectors for n-tuples:
Define (a1,a2, …, an) ∘(b1,b2,…, bn) to be the number a1b1+a2b2+… + anbn.

Unification: For any (a,b), **(a,b) ∘ (-b,a) = 0**
Generalization: For any (a1, a2,…,an), **(a1, a2,…,an) ∘ (-a2, a1, 0, … 0) = 0**

Examples of Abstraction

Abstraction of Addition: (Group, Ring, Field, Vector Space,…)
An Addition is a (commutative) binary operation "+" on set S for x and y in S to be the element of S, "x+y" with the property that x+y = y+x for any x and y in S.

Unification: For any a and b, (a,b)+(0,0) = (a,b)

Generalization: For any a1,a2, …, an, (a1,a2, …, an) +(0,0, …, 0) = (a1,a2, …, an)

Abstraction: "Additive identity" z in S is an additive identity: for any x in S, x + z = x.

Unification: For any (a,b), (a,b)+(-a,-b) = (0,0)

Generalization: For any (a1,a2, …, an) , (a1,a2, …, an) + (-a1,-a2, …, -an) = (0,0, … , 0)

Abstraction: "Additive inverse" For any x in S there is an element y of S with x + y = z .

Generalization of Inner Product of Vectors:
We say an operation "∘ " on a real vector space V is an inner product if for any v and w in V, v ∘w is a real number and the following properties are true: ……..

Unification: For any (a,b), (a,b) ∘ (-b,a) = 0

Generalization: For any (a1, a2,…,an), (a1, a2,…,an) ∘ (-a2, a1, 0, … 0) = 0

Abstraction: Suppose dim(V) >1, then for any v in V, there is a nonzero vector w in V with v ∘w = 0.

Examples for Understanding the Context of a Statement.

1. For a and b real numbers, $a \neq 0$, there is a unique real x with $ax=b$.
2. For a real number, $a \neq 0$, and an n-tuple of real numbers $(b1,b2,\ldots,bn)$ there is a unique n-tuple of real numbers $(x1,x2,\ldots,xn)$ with

   $$a\,(x1,x2,\ldots,xn) = (b1,b2,\ldots,bn).$$
3. For a scalar, $a \neq 0$, in the field F and a vector **b** in a vector space **V** over the field F, there is a unique vector **x** in **V** with $a\mathbf{x} = \mathbf{b}$.

Unification

1. For a and b real numbers, $a \neq 0$, there is a unique real x with $ax=b$.

- A unification in the context of the algebra of real numbers.
- Continue understanding by examining some cases, such as $a = 5$ and $b = 12$.
- Plan based on experience with algebra of real numbers.

Generalization

2. For a real number, $a \neq 0$, and an n-tuple of real numbers (b1,b2,…,bn) there is a unique n-tuple of real numbers (x1,x2,…,xn) with

   a (x1,x2,…,xn) = (b1,b2,…,bn).

- A generalization in the context of the algebra of real n- dimensional vectors in R^n .
- Continue understanding by examining some cases, n = 2,3, etc.
- Plan based on aspects of examples that generalize to all specified contexts.

Abstraction

3. For a scalar, $a \neq 0$, in the field F and a vector **b** in a vector space **V** over the field F, there is a unique vector **x** in **V** with a**x** = **b**.

- An abstraction in the context of the algebra of vectors in a vector space over a field.
- Continue understanding by examine some specific included contexts.
- Plan based on aspects of examples that rely only on abstract qualities of the examples.

Review of Changed Approach

Changed Approach to Proof Course

- Balanced Focus
  - **Polya 4 Phases**
  - **Context and organization**

1. **Understand** the context of the statement(s).
   - Is this a **unification,**
   - **generalization, or**
   - **abstraction?**
2. **R**ecognize how to use understanding of instances and the context to **develop a plan** with an appropriate organization (.
3. **Execute the plan** at the appropriate level of discourse and with a clear organization.

4. Reflect on the role of context and how that effected the plan and its execution.

References

Solow, Daniel. *The Keys to Advanced Mathematics: Recurrent Themes in Abstract Reasoning.* (Books Unlimited,1995) Now out of print.

Solow, Daniel. *How to Read and Do Proofs: An Introduction to Mathematical Thought Processes***,** 5th edition. (Wiley, 2009).

Pólya, George. *How to Solve It*. (Doubleday, 1957).

https://notendur.hi.is/hei2/teaching/Polya_HowToSolveIt.pdf

The End

- Questions?
- Comments?
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