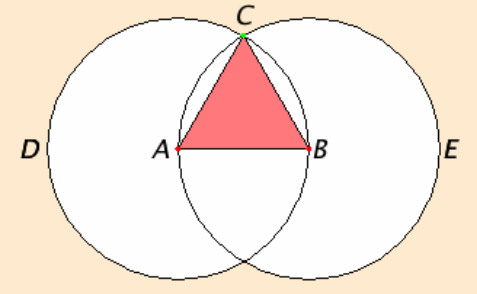# The Role of Philosophy in Proof: Euclid's Proof of Proposition One

Euclid
(325–265 BC)

Pi Mu Epsilon
Portland State University
Thursday, October 6, 2016

Martin Flashman

Humboldt State University

Arcata, CA

Preface

It is widely believed that logic is at the heart of proof in mathematics. Here is an alternative view that connects notions of proof with philosophical discussions related to ontology and epistemology.

Euclid's proof of Proposition 1 in his Elements, Book I, is offered as a primary example to illustrate some possible changes in focus.

Preparation Questions

How many of you have taken a course in Euclidean geometry with proofs?

How many of you have taken or are currently taking the transition to proof course? **Math 300 Introduction to Mathematical Reasoning**

How many of you have been taught "logic" with truth tables for propositions and venn diagrams for quantification?

How many of you have discussed some aspect of the philosophy of mathematics in your courses?

Polya's 4 Phases of Problem Solving

1. **Understand the problem.**

2. **See connections** to devise a plan.

3. Carry out the plan.

4. Look back. **Reflect on the process and results.**

Question: What role can philosophy play in problem solving and "proof".

Mathematics and Proof

Mathematical proof is not identical to a demonstration involving only truth tables and the syntax of quantification.

Material implication:

If the hypothesis of a conditional statement is false the statement is considered true.

Vacuous quantification:

If a statement is asserted about every member of an empty set the statement is considered true.

These two principles of logic are used in mathematics because **in mathematics the concern is on contexts where conditional statements and quantification have significant meaning**.

Philosophical Questions

What is the nature of the (mathematical) objects of discourse ? Ontology.

What is the nature of (mathematical) structures ? More Ontology.

What is the nature of (mathematical) knowledge ? Epistemology.

Ontology (Wikipedia)

- The philosophical study of the nature of being, becoming, existence, or reality, as well as the basic categories of being and their relations.

- What entities exist or can be said to exist, and how such entities can be grouped, related within a hierarchy, and subdivided according to similarities and differences.

Epistemology (Wikipedia)

- The branch of philosophy concerned with the nature and scope of knowledge and is also referred to as "theory of knowledge".

- What is knowledge?

- How can knowledge be acquired?

- To what extent can knowledge pertinent to any given subject or entity be acquired.

- How does the nature of knowledge relate to connected notions such as truth, belief, and justification?

Ontological Commitment

- In philosophy a "theory is **ontologically committed** (o.c.) to an object only if that object occurs in *all* the ontologies of that theory."

- Is geometry o.c. to points?

- Is arithmetic o.c. to numbers?

- Is mathematics o.c. to sets?

The Example(s) from Euclid

- **Euclid Book I Proposition 1**
  - *To construct an equilateral triangle on a given finite straight line.*
  - [*GeoGebra version*]

- Pythagoras (Euclid?):
  - *The square root of 3 is not a rational number.*

Euclid Book I Proposition 1

To construct an equilateral triangle on a given finite straight line.

Proof: Given finite straight line AB.

With center A construct circle O with radius AB.

With center B construct circle O' with radius AB.

Construct Segment AC from A to C, the point of intersection of O and O'.

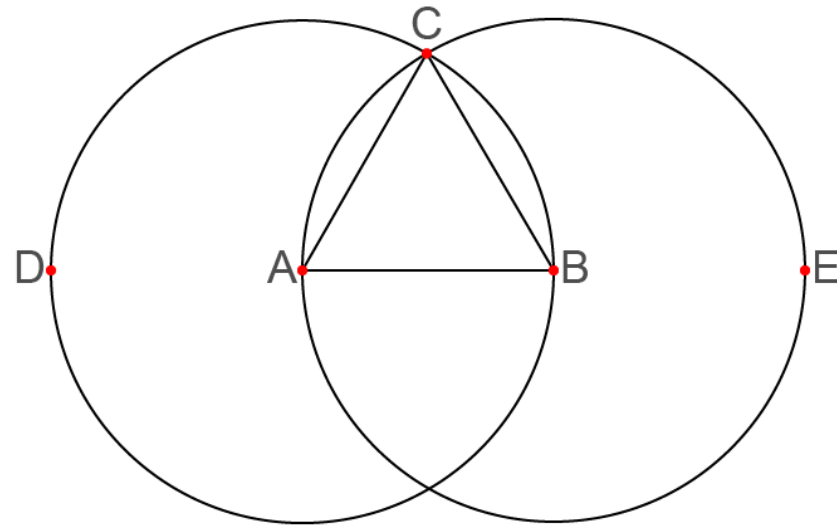Construct Segment BC from B to C, the point of intersection of O and O'.

AC = AB.

BC = AB.

The triangle ABC is the desired equilateral triangle.

QEF.

Euclid Book I Proposition 1

To construct an equilateral triangle on a given finite straight line.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interests:

- Construction is existence. QEF vs QED
- Definitions based on primitives: Point, Line, Circle
- Are Euclid Axioms built to model "reality"?
- Hilbert approach to (formal) axioms for geometry.

Euclid Book I Proposition 1

To construct an equilateral triangle on a given finite straight line.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interest – The Missing assumption:

The existence of point of intersection of circles.

- The power of counterexamples: Proofs and refutations (Lakatos)
- Alternative (models for) geometry :
  - Rational geometry.
  - Geometry without compass but with Playfair parallel postulate.

"Pythagoras"

The square root of 3 is not a rational number.

Proof:

Suppose r is a rational number and $r^2=3$.

r = a/b where a, b are positive natural numbers.

Then rb=a and $r^2 b^2 = a^2$. Or

$$3 b^2 = a^2.$$

Count **the number of 3 factors** of the right hand side: Even .

Count **the number of 3 factors** of the left hand side: Odd .

The Fundamental Theorem of Arithmetic (Euclid VII) applied to a^2 : The number of factors of a^2 and $3 b^2$ that are 3 is fixed].

Thus we have a contradiction and so the square root of 3 is not a rational number.

Q.E.D.

"Pythagoras"

The square root of 3 is not a rational number.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interests:

- Definitions of rationality emphasized.
  - Definitions and prior results in an information web. (Structures)
    - Equality of rational numbers and uniqueness of representation.
    - Prime number factors.
  - Equality and operations are context dependent.

"Pythagoras"

The square root of 3 is not a rational number.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interests:
- Algebraic Axioms or set theoretic construction of rational numbers. What is a fraction?
  - Existence usually ignored (presumed).
  - Existence usually presumed for numbers as measures from geometric model.
  - Numbers solve equations.
  - Existence of Algebraic and Transcendental Numbers:
    - What about the square root of -1.
    - What about π?

"Pythagoras"

The square root of 3 is not a rational number.

DISCUSSION---- What philosophical questions/issues does this proposition and proof pose?

Philosophical interests:

- Importance of consistency:
  - Mathematics abhors contradiction within its structures.
  - Indirect proof and indirect construction depend on consistency.

Euclid Definitions

- **Definition 1**.
  - A *point* is that which has no part.
- **Definition 2**.
  - A *line* is breadthless length.
- **Definition 3**.
  - The ends of a line are points.
- **Definition 4**.
  - A *straight line* is a line which lies evenly with the points on itself.
- **Definition 15**.
  - A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.
- **Definition 16**.
  - And the point is called the *center* of the circle.
- **Definition 19**.
  - *Rectilinear figures* are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.
- **Definition 20**.
  - Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.

Euclid Postulates

- Let the following be postulated:
- Postulate 1.
    - **To draw a straight line from any point to any point.**

- Postulate 2.
    - **To produce a finite straight line continuously in a straight line.**

- Postulate 3.
    - **To describe a circle with any center and radius.**

- Postulate 4.
    - That all right angles equal one another.

- Postulate 5.
    - That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

A Model for Geometry: Rational Geometry.

- **What is <u>a point</u>?** An ordered pair of **rational numbers. (a,b).**

- **What is <u>a line</u>? A set of points where the coordinates (x,y) of each point in the set satisfy an equation Ax +By = C where A,B and C are rational numbers – not all 0.**

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A Model for Geometry: Rational Geometry.

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- **What is a line? A set of points where the coordinates (x,y) of each point in the set satisfy an equation Ax +By = C where A,B and C are rational numbers – not all 0.**

- **What is <u>a circle</u>? A set of points** where the coordinates (x,y) satisfy an equation **(x-a)^2 + (y-b)^2 = c** where a,b and c are rational numbers – and c > 0.

- **Notice- x and y must be rational numbers**

Why is this a "model" for Geometry

- Using these definitions to characterize what is a point, the postulates for Euclid's geometry are satisfied:
- Two points determine a (unique) line:
- E.g.(3,5) and (2,7) determine the line with equation: $2(x-3) + (y-5) = 0$.
- Two points determine two circles:
- E.g.(3,5) and (2,7) determine the circles with equations:
- **$(x-3)^2 + (y-5)^2 = 5$ and**
- **$(x-2)^2 + (y-7)^2 = 5$.**

Why Proposition 1 and it's proof fails in Rational Geometry

- Consider if Proposition 1 is true in Rational Geometry. Then the pythagorean theorem can be proven in this geometry.
- Now consider the segment determined by the points (0,0) and (2,0). Certainly the two circles

$$x^2 + y^2 = 4 \text{ and } (x-2)^2 + y^2 = 4$$

- **are the equations of the circles indicated by the proof of Proposition 1. But if a point (x,y) lies on both these circles, then**
- **$x^2 = (x-2)^2$ and so x=1.**
- **But then $1^2 + y^2 = 4$ and $y^2 = 3$.**

Why Proposition 1 and it's proof fails in Rational Geometry

- So- if Proposition 1 is true in Rational Geometry. **a point (x,y) that lies on both the circles determined by the points $(0,0)$ and $(2,0)$ must have $x = 1$ and $y^2 = 3$.**

- **But then y would be a rational number with $y^2 = 3$ which contradicts Pythagoras' Theorem:** The square root of 3 is not a rational number.

- So … **there is no such point in rational geometry and there is no equilateral triangle in this geometry on the segment determined by the points $(0,0)$ and $(2,0)$.**

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The Main Question:

Should Philosophy Play a Larger Role in Understanding Proofs?

Answer:

?

The End

- Thanks-
- Questions

These slides will be posted at
http:// users.humboldt.edu/flashman