What can we learn from Newton's estimate of ln(2) and Pi?

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Abstract

Newton made a very accurate estimate for the hyperbolic logarithm of 2 by combining understanding of properties of logarithms, the geometric series, and integration for polynomials.

The author will analyze Newton's approach and explore how this approach might be better understood by students by asking for an estimate of pi using the fact that

$$\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\frac{\pi}{4}.$$

Outline

- I. Analyze **the approach used by Isaac Newton in his 16 decimal place estimate for the natural logarithm of 2 which appeared in "*The method of fluxions and infinite series*" [1671/1736]**
- II. Examine briefly **Newton's estimate of π** in the same publication to see how it follows a somewhat similar approach.
- III. Follow Newton's approach from the logarithm more schematically by asking for an estimate of π using the fact

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}.$$

Part I. Newton's computations of hyperbolic logarithms.

- In 1676 Newton wrote in a letter to Henry Oldenburg on some of his applications of series to estimating areas, in particular in estimating areas for the hyperbolic logarithm.
- This work was later clarified in *Of the Method of Fluxions and Infinite Series* which was published posthumously in 1737, ten years after Newton's death.

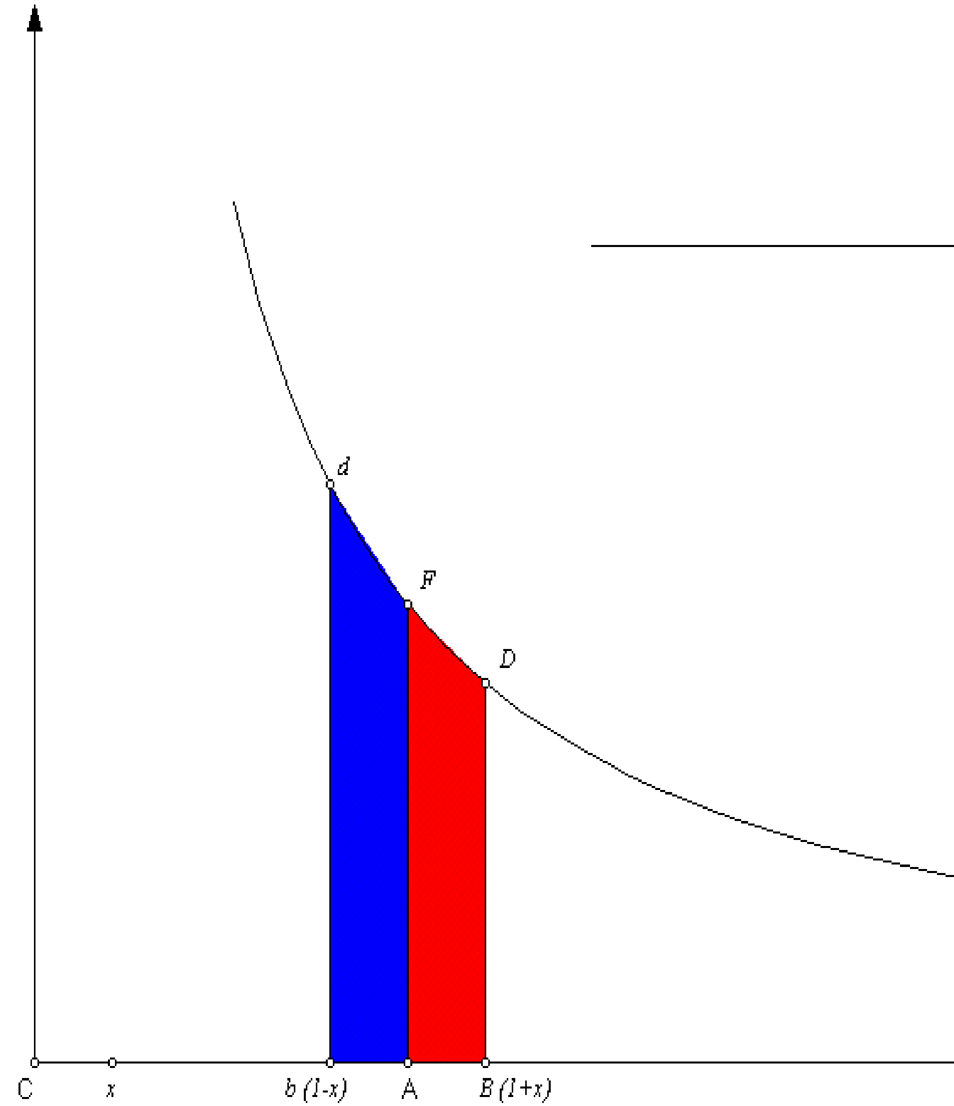
Newton estimates the Hyperbolic Log

Newton considers symmetrically located points on the main axis, $1 + x \ \ and \ \ 1 - x$ with $x > 0$ and their related reciprocals.

He then uses two integrals related to the geometric series to determine the related areas,

(i) between the hyperbola and above the segment $[1, 1 + x]$ (red) AFDB and

(ii) between the hyperbola and above the segment $[1 - x, 1]$ (blue) AFdb.



Newton estimates the Hyperbolic Log

The integrals to determine the related areas,

(i) between the hyperbola and above the segment $[1,1+h]$ (red) AFDB

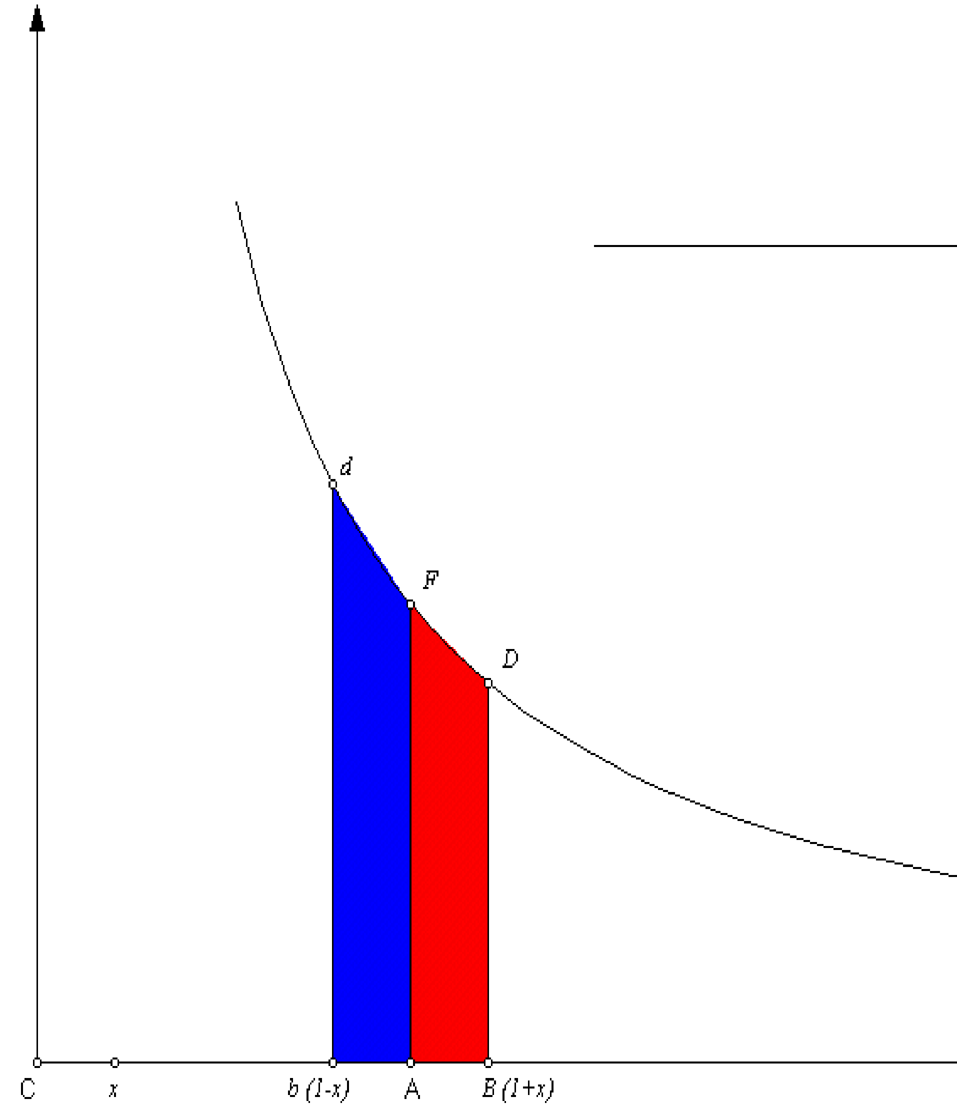
$$\int_1^{1+h} \frac{1}{x}\,dx = \int_0^h \frac{1}{1+x}\,dx$$

$$[u = x + 1]$$

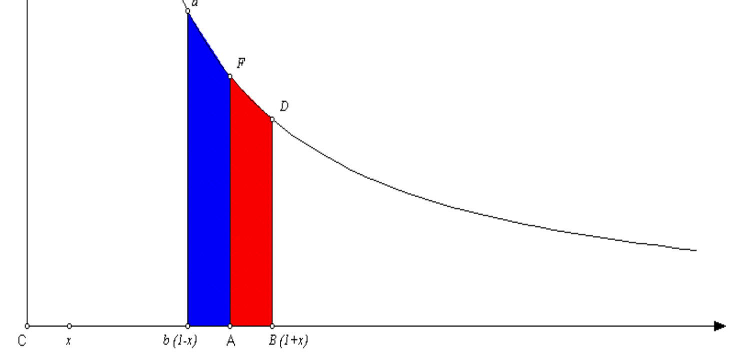
(ii) between the hyperbola and above the segment $[1-h\,,1]$ (blue) Afdb

$$\int_{1-h}^{1} \frac{1}{x}\,dx = \int_0^h \frac{1}{1-x}\,dx$$

$$[u = 1 - x]$$



The red and the blue.



$$Area\ AFDB = \int_0^h \frac{1}{1+x}\,dx = \int_0^h 1 - x + x^2 - x^3 + ... = h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} ....$$

$$Area\ AFdb = \int_0^h \frac{1}{1-x}\,dx = \int_0^h 1 + x + x^2 + ... + x^k + .. = h + \frac{h^2}{2} + \frac{h^3}{3} + ... \frac{h^k}{k} + ...$$

These allow the estimation of the sum and difference of the two areas:

$$Total\ area\ bdDB = 2h + 2\frac{h^3}{3} + 2\frac{h^5}{5} + 2\frac{h^7}{7} + ...$$

$$Difference\ of\ areas\ Ad - AD = h^2 + \frac{h^4}{2} + \frac{h^6}{3} + \frac{h^8}{4} + ...$$

Now to find the Area of the two separate regions (and related logarithms) we take $1/2$ of the difference of these results and $1/2$ of the sum of these results.

- Newton uses the first eight terms with

$h = .1$ and $h = .2$ to estimate $the\ hyperbolic$ ($natural$) logarithm of

$$0.9, 1.1, 0.8 \ and\ 1.2$$

Sum of Areas

h	0.1	.2
2h	0.2	0.4
$2h^3/3$	0.000666666666666	0.00533333333333
$2h^5/5$	0.000004	0.000128
$2h^7/7$	0.0000000285714286	0.00000365714285714
$2h^9/9$	0.0000000002222222	0.00000011377777778
$2h^{11}/11$	0.0000000000018182	0.00000000372363636
$2h^{13}/13$	0.0000000000000154	0.00000000012603077
$2h^{15}/15$	0.0000000000000001	0.00000000000436907
Sum of Areas	0.200670695462151	0.405465108108002

Difference of Areas

h	0.1	.2
h^2	0.01	0.04
$h^4/2$	0.00005	0.0008
$h^6/3$	0.0000003333333333	0.0000213333333
$h^8/4$	0.0000000025	0.00000064
$h^{10}/5$	0.00000000002	0.0000002048
$h^{12}/6$	0.0000000000001667	0.00000000068267
$h^{14}/7$	0.0000000000000014	0.0000000002341
Diff'ce of Areas	0.010050335853501	0.040821994519406

The Area of the two separate regions(and related logarithms)

1/2 of the difference of these results and 1/2 of the sum of the results.

ln(1.1)≈1/2 (0.2006706954621511- 0.0100503358535014)

≈ 0.0953101798043248

ln(.9) ≈ -(1/2)(0.2006706954621511 + 0.0100503358535014)

≈ -0.105360516578263 .

ln(1.2) ≈ 1/2 (0.405465108108002 -0.040821994519406)

≈ 0.1823215557693954 6 (from Newton)

ln(.8) ≈ -(1/2)(0.405465108108002 +0.040821994519406)

≈ -0.2231435513142097 (from Newton) .

Final calculations for ln(2)

$$\ln(2) = \ln\left(\frac{1.2}{.8}\frac{1.2}{.9}\right) = 2\ln(1.2) - (\ln(.9) + \ln(.8))$$

$$\approx 2(0.18232155576939546)$$
$$+ 0.10536051657826 3 + \ 0.22314355131420 97$$

$$= \ 0.6931471805599453\ (from\ Newton)$$

Comparison

ln(2) from Newton:

0.6931471805599453

ln(2) from calculator:

0.69314718055994530941723212145818

From Newton

From Newton, *Of the Method of Fluxions and Infinite Series* , pp 132-133.

Newton_on PI.pdf

ing these numbers for a, b, and x, the first term of the series becomes 0.2, the second 0.0006666666666666, &c. the third 0.000004, and so on; as you see in this table.

```
0.2000000000000000
      666666666666
       40000000000
         285714286
           2222222
             18182
               154
                 1
------------------
0.2006706954621511 = Area bdDB.
```

If the parts of this Area Ad and AD be added separately, subtract the lesser DA from the greater dA, and there will remain $\frac{bx^2}{a}+\frac{bx^4}{2a^3}+\frac{bx^6}{3a^5}+\frac{bx^8}{4a^7}$, &c. where, if 1 be wrote for a and b, and $\frac{1}{10}$ for x, the terms being reduced to decimals will stand thus.

```
0.0100000000000000
      500000000000
       33333333333
          25000000
            200000
              1667
                14
------------------
0.0100503358535014 = Ad—AD.
```

Now if this difference of the Areas be added to, and subtracted from, their sum before found; half the aggregate 0.1053605156578263 will be the greater

greater Area Ad; and half the remainder 0.0953101798043248 will be the lesser Area AD.

By the same tables these Areas AD and Ad will be obtained also, when AB and Ab are supposed $\frac{1}{100}$, or CB = 1.01, and Cb = 0.99; if the numbers are but duly transferred to lower places. As

```
    0.0200000000000000          0.0001000000000000
          666666666666                    50000000
               4000000                        3333
                    28          ------------------
    ------------------          0.0001000050003333 = Ad—AD
Sum 0.0200006667066695 = bD
```

Half the aggregate 0.0100503358535014 = Ad and Half the residue 0.0099503308531681 = AD.

And so putting AB and Ab = $\frac{1}{1000}$, or CB = 1.001, and Cb = 0.999, there will be obtained Ad = 0.0010005000333583 5 and AD = 0.0009 995001333083 5.

In the same manner (if CA and AF = 1) putting AB and Ab = 0.2, or 0.02, or 0.002, these areas will arise.

Ad = 0.2231435513142097 and AD = 0.1823215576939546
or Ad = 0.0202027073175194 and AD = 0.0198026272961797
or Ad = 0.002002 and AD = 0.001

From these Areas thus found it will be easy to derive others by addition and subtraction alone, for as it is $\frac{1.2}{0.8} \times \frac{1.2}{0.9} = 2$; the sum of the areas 0.6931471805599453 belonging to the ratios $\frac{1.2}{0.8}$ and $\frac{1.2}{0.9}$ (that is insisting upon the parts of the abscis 1.2, 0.8. and 1.2, 0.9.) will be the area AF$\delta\beta$, when Cβ = 2; as is known. Again, since $\frac{1.2}{0.8}$; $\times$ 2 = 3, the sum 1.0986122886681097 of

Summary Analysis of Computation

1. Use of geometric "series and polynomials" to estimate $\frac{1}{1+x}$ and $\frac{1}{1-x}$ $when\ x\ \approx 0$.
2. Integration of polynomials.
3. Geometry and algebra to decompose and recover estimates.
4. Algebra of logarithmic function.

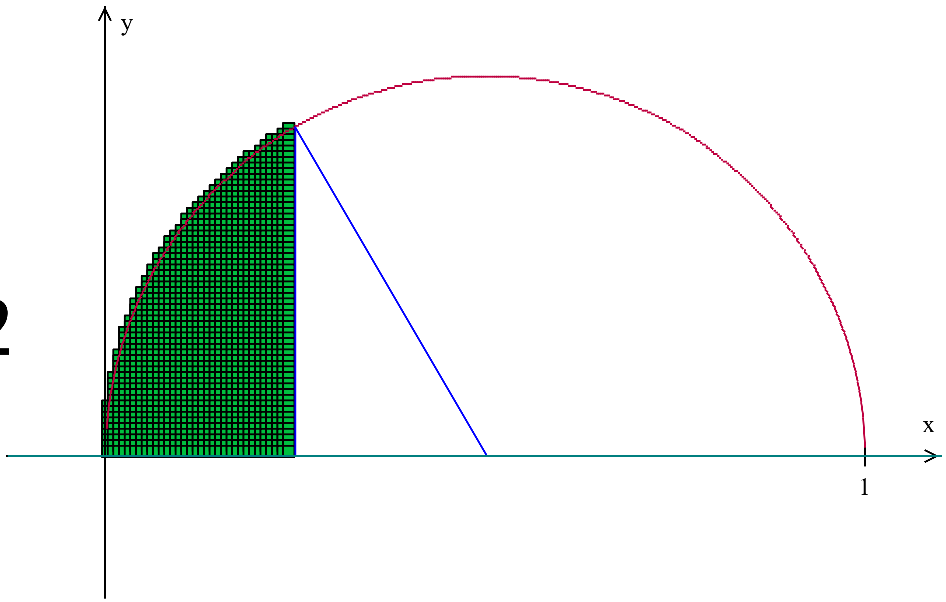
$$\ln(\frac{A^2}{C*D})\ = 2\ln(A) - (\ln(C)\ + \ln(D)).$$

Part II. Newton's computations of “π”

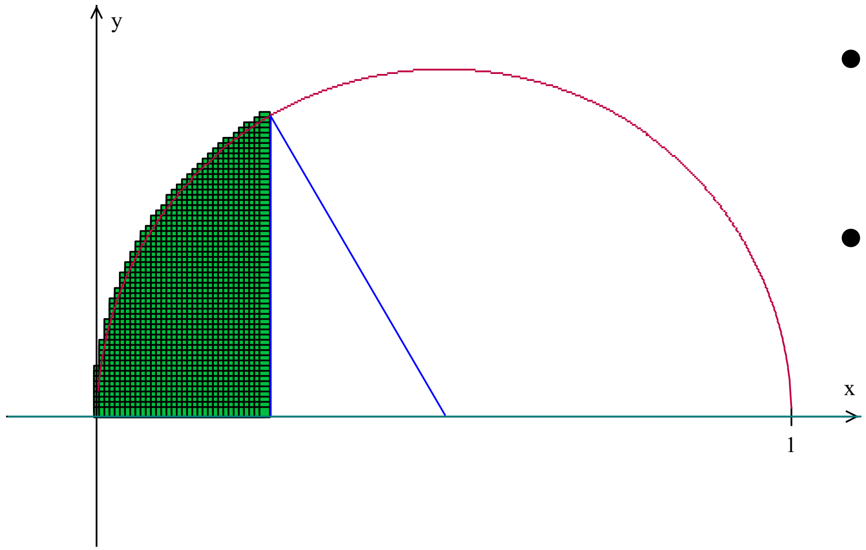
- Circumference of a circle is “$2\pi r$”.
- Area of a circle is “πr^2”.
- **Locate circle** of radius 1/2 with center at (1/2,0).
- **Equation for circle** is

$$y^2 = x\,(1 - x)$$

$$y = \sqrt{x}\,\sqrt{1-x}$$



Newton's Estimate of "π"



- Use "series and polynomials" to estimate $y = \sqrt{x}\sqrt{1-x}$.
- Integrate "polynomials" to estimate area from 0 to 1/4.
- Combine the area of the triangle from 1/4 to ½ to $(1/4, \sqrt{3}/4)$ with the shaded area under the circle from $0\ to\ 1/4$ to cover the area of the central sector of 1/6th of circle. This gives an estimate of "$\pi/24$" to 16 places!

Use of "Polynomials" and Integration

Polynomials used for $\sqrt{1-x}$:(Binomial Series)

$$\sqrt{1-x}: 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \ldots$$

$$\sqrt{x}\,\sqrt{1-x}: x^{1/2} - \frac{x^{\frac{3}{2}}}{2} - \frac{x^{\frac{5}{2}}}{8} - \frac{x^{\frac{7}{2}}}{16} - \frac{5x^{\frac{9}{2}}}{128} - \frac{7x^{\frac{11}{2}}}{256} \ldots$$

Now integrate to obtain:

$$\frac{2x^{\frac{3}{2}}}{3} - \frac{x^{\frac{5}{2}}}{5} - \frac{x^{\frac{7}{2}}}{28} - \frac{x^{\frac{9}{2}}}{72} - \frac{5x^{\frac{11}{2}}}{704} - \frac{7x^{\frac{13}{2}}}{1664} \ldots$$

And for area of region under circle evaluate at ¼ :

$$\frac{2(\frac{1}{2})^3}{3} - \frac{(\frac{1}{2})^5}{5} - \frac{(\frac{1}{2})^7}{28} - \frac{(\frac{1}{2})^9}{72} - \frac{5(\frac{1}{2})^{11}}{704} - \frac{7(\frac{1}{2})^{13}}{1664} = \frac{1}{12} - \frac{1}{160} \ldots$$

Finale for Newton's estimate of "π"

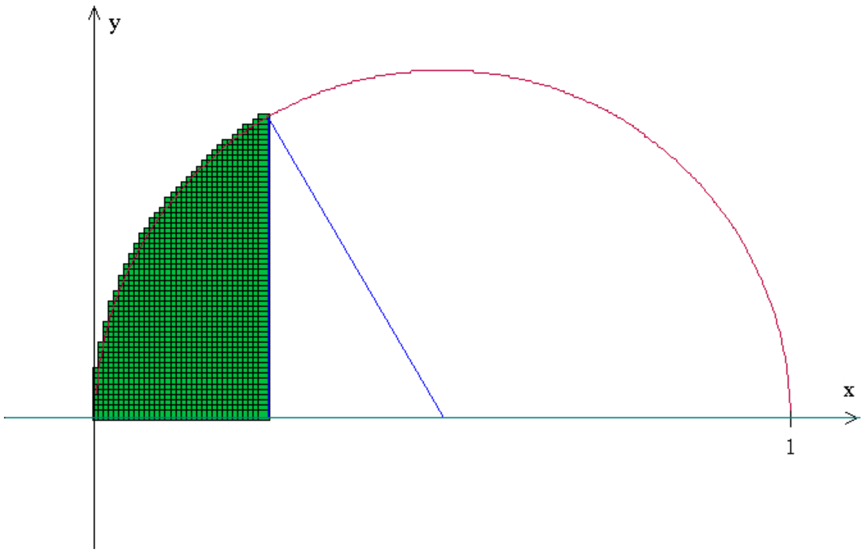
Area of Triangle: $\frac{\sqrt{3}}{32}$.

Area of central sector:

$\frac{1}{6}th$ of circle of radius $\frac{1}{2}$

Area of circle (radius $\frac{1}{2}$) ≈

$$6\left(\frac{\sqrt{3}}{32}+\frac{1}{12}-\frac{1}{160}-\ \dots.\right)$$



Circumference of circle (radius $\frac{1}{2}$) $= 4\text{Area}\ (= \pi)$

≈**(Newton)**(4)[$6\left(\frac{\sqrt{3}}{32}+\frac{1}{12}-\frac{1}{160}-\ \dots.\right)$]=

3.1415926535897**928**

Comparison

π from Newton:

3.1415926535897**28**

π from calculator:

3.14159265358979**323846 26433832795**

From Newton

From Newton, *Of the Method of Fluxions and Infinite Series* , pp 130-131.

Newton_on PI.pdf

the terms thus reduced by degrees, I difpofe into Two Tables; the affirmative terms in One, and the Negative in Another, and add them up as you fee here.

+ 0.0833333333333333	— 0.0002790178571429
625000000000000	34679066051
271267361111	834465027
5135169396	26285354
144628917	961296
4954581	38676
190948	1663
7963	75
352	4
16	0.0002825719389575
1	+ 0.0896109885646618
0.0896109885646618	0.0893284166257043

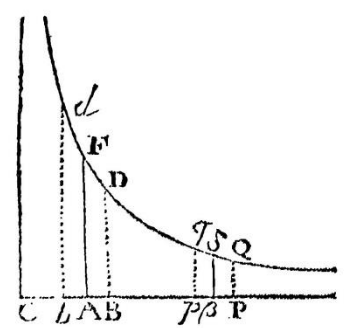
Then from the fum of the affirmative, I take the fum of the negative terms, and there remains 0.0893284166257043 for the quantity of the Hyperbolick Area A*d*B which was to be found.

Let the Circle A*d*F [*See the fame Fig.*] be propofed, which is expreffed by the equation $\sqrt{x-xx}=\dot{z}$, whofe diameter is unity; and from what goes before its Area A*d*B will be $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}$, &c. in which feries, fince the terms do not differ from the terms of the feries which above expreffed the Hyperbolick Area, except in the figns + and —; nothing elfe remains to be done, than to connect the fame numeral terms with their figns; that is, by fubtracting the connected fums of both the forementioned Tables, 0.0898935605036193, from the firft term doubled 0.1666666666666666, &c. and the remainder 0.0767731061630473 will be the portion A*d*B of the Circular Area, fuppofing AB to be a fourth part of the Diameter. And hence we may obferve, that though the Areas of the Circle and Hyperbola are not expreffed in a Geometrical confideration, yet each of them is difcovered by the fame Arithmetical computation.

The portion of the Circle A*d*B being found, from thence the whole Area may be derived. For the radius *d*C being drawn, multiply B*d* or $\frac{1}{4}\sqrt{3}$ into BC or $\frac{1}{4}$, and one half of the product $\frac{1}{32}\sqrt{3}$, or 0.0541265877365275 will be the value of the Triangle C*d*B; which added to the Area A*d*B, will give the Sector AC*d*, 0.1308996938995747; the Sextuple of which 0.7853981633974482 is the whole Area.

And hence (by the way) the length of the Circumference will be 3.1415926535897928, which is found by dividing the Area by a fourth part of the diameter.

To this we fhall add the calculation of the Area comprehended between the Hyperbola *d*FD and its Afymptote CA, let C be the center of the Hyperbola, and putting CA $=a$, AF $=b$, and AB $=$ A*b* $=x$; it will be $\frac{ab}{a+x}=$ BD, and $\frac{ab}{a-x}=bd$; whence the Area AFDB $=bx-\frac{bxx}{2a}+\frac{bx^3}{3a^2}-\frac{bx^4}{4a^3}$, &c. And the



Area AF*db* $=bx+\frac{bx^2}{2a}+\frac{bx^3}{3a^2}+\frac{bx^4}{4a^3}$, &c. And the fum *bd*DB $=2bx+\frac{2bx^3}{3a^2}+\frac{2bx^5}{5a^4}+\frac{2bx^7}{7a^6}$, &c. Now let us fuppofe CA $=$ AF $=1$, and A*b* or AB $=\frac{1}{10}$, C*b* being $=0.9$, and CB $=1.1$. then fubftitut-

S 2 ing

7905	3
352	4
16	
1	0.0002825719389575
	+ 0.0896109885646618
0.0896109885646618	0.0893284166257043

Then from the ſum of the affirmative, I take the ſum of the negative terms, and there remains 0.0893284166257043 for the quantity of the Hyperbolick Area AdB which was to be found.

Let the Circle AdF [*See the ſame Fig.*] be propoſed, which is expreſſed by the equation $\sqrt{x-xx}=z$, whoſe diameter is unity; and from what goes before its Area AdB will be $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}$, *&c.* in which ſeries, ſince the terms do not differ from the terms of the ſeries which above expreſſed the Hyperbolick Area, except in the ſigns + and —; nothing elſe remains to be done, than to connect the ſame numeral terms with their ſigns; that is, by ſubtracting the connected ſums of both the forementioned Tables, 0.0898935605036193, from the firſt term doubled 0.1666666666666666, *&c.* and the remainder 0.0767731061630473 will be the portion AdB of the Circular Area, ſuppoſing AB to be a fourth part

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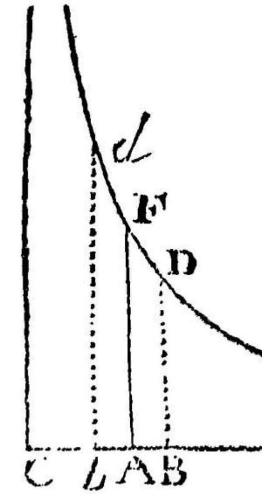

Area AF$db=bx+\frac{bx^2}{2a}+\frac{bx^3}{3a^2}+\frac{bx^4}{4a^3}$…

ſum bdDB$=2bx+\frac{2bx^3}{3a^2}+\frac{2bx^5}{5a^4}+\frac{2bx^7}{7a^6}$…

let us ſuppoſe CA$=$AF$=1$, and A… Cb being $=0.9$, and CB$=1.1$. th…

S 2

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333333333333	—0.0002790178571429
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109885646618	
	0.0893284166257043

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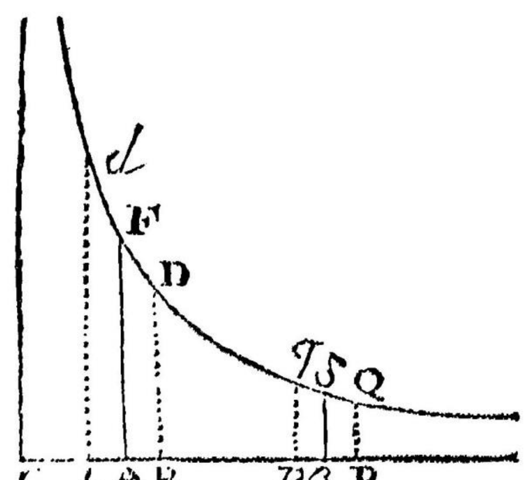


Analysis of Computation

1. Use of binomial series polynomials to estimate $\sqrt{1-x}$ Then multiplied by $\sqrt{x}$.
2. Integration of “series … polynomials”.
3. Geometry and algebra to decompose and recover estimates.
4. Algebra of geometric areas:

Area of sector = area of triangle + area under circle.

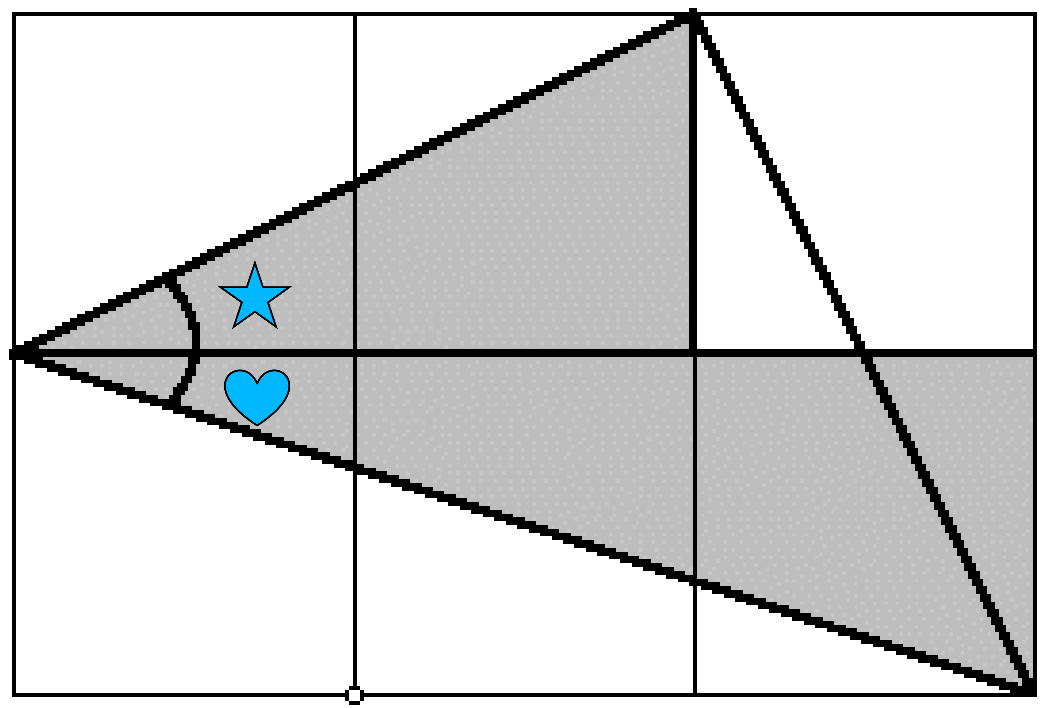
Part III.
Newton's logarithmic scheme for computations applied to estimating "π" .

Basic Scheme:

- Use polynomials as geometric series for $\frac{1}{1+x^2}$.
- Integration of $\frac{1}{1+x^2}$ polynomials gives polynomial for arctan(x).
- Use values "close to 0."
  - arctan(1/2); arctan(1/3)
- Use addition reductions.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}.$$

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$$



Note on Other Estimates of π

John Machin (1706-**Jones**) : 100 places

William Shanks (1873): 707 places- 527 correct!

$$4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) = \frac{\pi}{4}.$$

Euler: adopted π as a symbol.

Developed other equations for estimation such as

$$5\arctan\left(\frac{1}{7}\right) + 2\arctan\left(\frac{3}{79}\right) = \frac{\pi}{4}$$

Current best (?): Alexander J. Yee & Shigeru Kondo (August 2, 2010) 5 trillion places.

Exercise for Calculus

Apply the analysis from Newton’s estimate for ln(2) to create an estimate for “π” from the arctan identity…

Estimate of “π” with Excel

k	n=2k-1	(-1)^(k+1)x^n/n x=1/2	(-1)^(k+1)x^n/n x=1/3
1	1	0.50000000000000000000	0.33333333333333000000
2	3	-0.04166666666666700000	-0.01234567901234570000
3	5	0.00625000000000000000	0.00082304526748971200
4	7	-0.00111607142857143000	-0.00006532105297537400
5	9	0.00021701388888889000	0.00000564502926947676
6	11	-0.00004438920454545450	-0.00000051318447904334
7	13	0.00000939002403846154	0.00000004824811341433
8	15	-0.00000203450520833333	-0.00000000464611462508
9	17	0.00000044878791360294	0.00000000045550143383
10	19	-0.00000010038677014803	-0.00000000004528376828
11	21	0.00000002270653134301	0.00000000000455233649
12	23	-0.00000000518301258916	-0.00000000000046183124
13	25	0.00000000119209289551	0.00000000000004720942
14	27	-0.00000000027594742952	-0.00000000000000485694
15	29	0.00000000006422914308	0.00000000000000050244
	Estimate	**0.46364760901297200000**	**0.32175055439664200000**
	arctangent	**0.46364760900080600000**	**0.32175055439664200000**
		pi estimate = 4(atan(1/2)+atan(1/3))	**3.14159265363846**
		pi	**3.14159265358979**

The End
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Questions?

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Miscellaneous

Another reference: Logarithms: The Early History of a Familiar Function

by Kathleen M. Clark (Florida State University) and Clemency Montelle (University of Canterbury)

http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=3495&bodyId=3845