Making Sense of Series and Sequences in The First Calculus Course. Preliminary report.

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What's Happening Now in The First Calculus Course

- Differential and Integral Calculus with a variety of theory and applications. THEN...
- Sequences and Sums of Numbers: the Theory and Tests of Convergence
  - Geometric sequences and series.
  - Positive Series Tests [Monotonic, Comparison, Integral, …]
  - Alternating Series [Absolute, Conditional, ...]
  - The Ratio and Root Tests.

What's Happening Now in The First Calculus Course

- Power series
  - Intervals of Convergence
  - Functions Defined from Series.
  - Diff'n/Int'gr'n: Calculus of Power Series
  - **Taylor / MacLaurin Series and Theorem**
  - Series expansion for exp, ln, trig, binomial, etc

What's Happening Now in The First Calculus Course

- Applications: Series used for
  - Estimation of Numbers: e, pi, sqrt,...
  - Estimation of Definite Integrals.
  - Solution of Differential Equations.

What's Happening Now in The First Calculus Course

Critique

- **Little motivation from previous work**
  - **Newton's method.**
  - **Estimates of Definite Integrals.**
  - **Solutions to Differential Equations.**
- **Delayed Connection with the previous progress**
- **Unclear statement of what is fundamental.**

Making Sense of The First Calculus Course

- Experience- Spring 1978, 1979

  Two years of teaching 2nd semester of Calculus with students who had 1st semester with another instructor.
- Need for a better approach for all of first year.

Making Sense of The First Calculus Course

"The key to solving the 'calculus problem'.... make sense in our calculus instruction ....

internally and in context,

to ourselves as instructors and

to our students as learners.

This criterion will provide the knife for cutting and the thread for reassembling the calculus curriculum of the next 40 to 50 years."

Quote was from the Editorial that appeared in The UMAP Journal in 1990.

"A Sensible Calculus"

Making Sense of The First Calculus Course

Three themes...for reviewing and revising the calculus curriculum, namely,

- **Differential Equations,**
- **Estimation, and**
- **Mathematical Modeling**

The Sensible Calculus Program

- My solution: (now with Tami Matsumoto of College of the Redwoods)
  http://users.humboldt.edu/flashman/senscalca_x.html
- Search: Sensible calculus

The Sensible Calculus Program Overview

- The calculus of Taylor polynomials (not series) appears as **a tool for approximating difficult definite integrals with a sensible control on the error.**
- **Estimating the solution to differential equations such as y" = -y or y"= y with y(0) = 1 and y'(0) = 1** provides additional motivation for the convergence questions of infinite series.

The Sensible Calculus Program Overview

- **Infinite** sequences and series analysis discuss Taylor theory examples from the beginning along with the traditional examples of geometric and harmonic series.
- Historical connections Newton's work in estimating the values of the natural logarithm illustrate how geometric series played a significant role in showing the power of the early calculus for computation and estimation.

Start with Connections

- Review of previous work on estimates that have been sequential.
- Estimates for integrals:
  - constant,
  - Linear,
  - quadratic,
  - interpolation (trapezoid, simpson) versus single point based on derivative (midpoint).
- Ease of polynomials for derivatives, integrals, and computation.

From Sensible Calculus

Taylor Theory and Series:

- **Taylor Theory without series.**
- Example: Section IX.A introducing Taylor Theory for e^x with applications.

http://users.humboldt.edu/flashman/book/ch9/IXA.htm

Taylor Theory-Objective & Key Ideas

Two Objectives:

- Find estimating polynomials for a given function

- Measure the error in using polynomials to estimate.

Taylor Theory-Objective & Key Ideas

Two Key Ideas:

- When x is close to a, f(x) is approximately equal to a linear function, **f(a) + f '(a)(x-a).**
- As long as f is a sufficiently well behaved function there is some c between a and x where

$$\mathbf{f(x) = f(a) + f'(c)(x-a).}$$

Focus Themes for Series: Estimation, Differential Equations, Models

- First Section! Focus on growth model & differential equation: $P'(x) = P(x), P(0) = 1.$
- Solution is already treated with estimation by Euler's method.
- Solution is “known”: $P(x) = e^x$.
- Estimation: Use polynomial of degree n that best matches the differential equation at x= 0.
- Determine the error for estimating e and

$$\int_0^1 e^{-x^2} dx$$

Sensible Sequences & Series

- IX.A Taylor Theory for e^x
- IX.B MacLaurin Polynomials and Taylor
- IX.C MacLaurin Polynomials: How to Find Them
- IX.D Taylor Polynomials

Sensible Sequences & Series

- X.A Simple Examples and Definitions: Visualizing Sequences.
- X.B Series: Key Examples and Results.
  - Geometric series
  - Harmonic series
  - Alternating Harmonic series
  - MacLaurin - Taylor series
- X.B5 Tests for Divergence and Convergence.
  - Testing aimed at theme- power series and DE's from models.
  - Key examples: e^x , Taylor series and Geometric series.

Power Series: A Thematic Sensible Conclusion

- The Interval and Radius of Convergence.
- Functions and power series.
- $P(x) = \Sigma\, a_n x^n$
- Is P a differentiable function? What is the derivative of P? Is P a C^∞ function?
- What are the Taylor polynomials and the Taylor series for P?
- What is the relation of functions defined by power series to the problem of solving differential equations?

Conclusion

- With this reorganization, the treatment of sequences and series forms **a sensible part of the first year calculus program,** completing an approach that focuses on understanding the three mathematical themes:
  - **Differential Equations,**
  - **Estimation, and**
  - **Mathematical Modeling.**

The End.

Questions

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Link: The Sensible Calculus Program

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/senscalca_x.html