Loughborough University
Mathematics Education Centre
Culture, Pedagogy and Identity Interest Group

Making sense of solving equations visually using functions and mapping diagrams

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- Mapping diagrams provide a valuable and underused tool for visualizing functions that can connect function concepts to solving equations in many contexts.
- In this presentation I will use mapping diagrams and GeoGebra to make sense visually

i. of linear and quadratic real functions
ii. of steps used in common algebraic approaches to solving equations and
iii. complex numbers and functions.

Making sense of solving equations visually using functions and mapping diagrams

Links:

http://users.humboldt.edu/flashman/Presentations/MEC2017/MEC.MD.LINKS.html

Mapping Diagram Sheets	Mapping Diagram blanks (2 axis diagrams)	Mapping Diagram blanks (2 and 3 axes)
Work/Spreadsheets	Worksheet.pdf	Spreadsheet Template (Linear Functions)
Section from MD from A B to C and DE (Drafts)	Visualizing Functions: An Overview	Linear Functions (LF) Quadratic Functions(QF)
GeoGebra	Visualize Solving a Linear Equation using Mapping Diagrams	Mapping Diagrams for Solving a Quadratic Equation
YouTube Videos	Using Mapping Diagrams to Visualize Linear Functions (10 Minutes)	Solving Linear Equations Visualized with Mapping Diagrams. (10 Minutes)

Background Questions

- Are you familiar with Mapping Diagrams to visualize functions?
- Have you used or experienced Mapping Diagrams to teach functions?
- Have you used or experienced Mapping Diagrams to teach content besides function definitions?

Main Resource

- Mapping Diagrams from A(lgebra) B(asics) to C(alculus) and D(ifferential) E(quation)s. A Reference and Resource Book on Function Visualizations Using Mapping Diagrams (Preliminary Version)

- http://users.humboldt.edu/flashman/MD/section-1.1VF.html

Mapping Diagram Prelim

- Examples of mapping diagrams
  - Worksheet 1.a
  - Make tables for $m(x) = 2x$ and $s(x) = x+1$

x	m(x) =2x
2	
1	
0	
-1	
-2	

x	s(x) =x+1
2	
1	
0	
-1	
-2	

Mapping Diagram Prelim

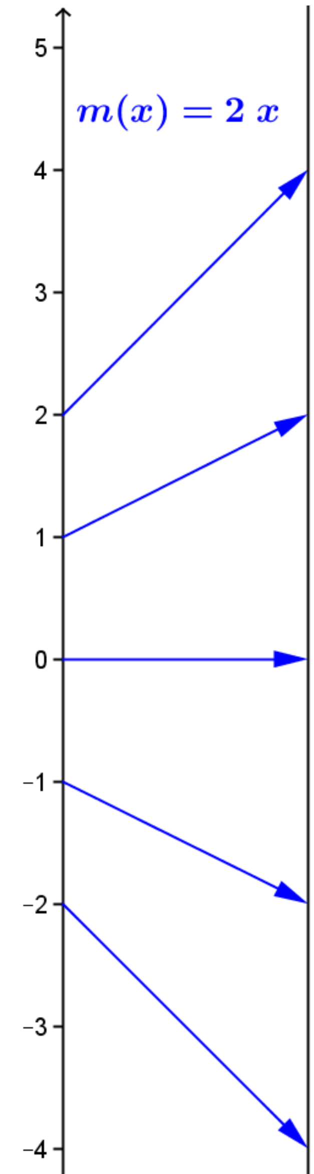
- Examples of mapping diagrams
  - Worksheet 1.b
  - On separate diagrams sketch mapping diagrams for m(x) = 2x and s(x)= x+1

x	m(x) =2x
2	4
1	2
0	0
-1	-2
-2	-4

x	s(x) =x+1
2	3
1	2
0	1
-1	0
-2	-1

Worksheet 1.b Mapping Diagram: m(x) = 2x

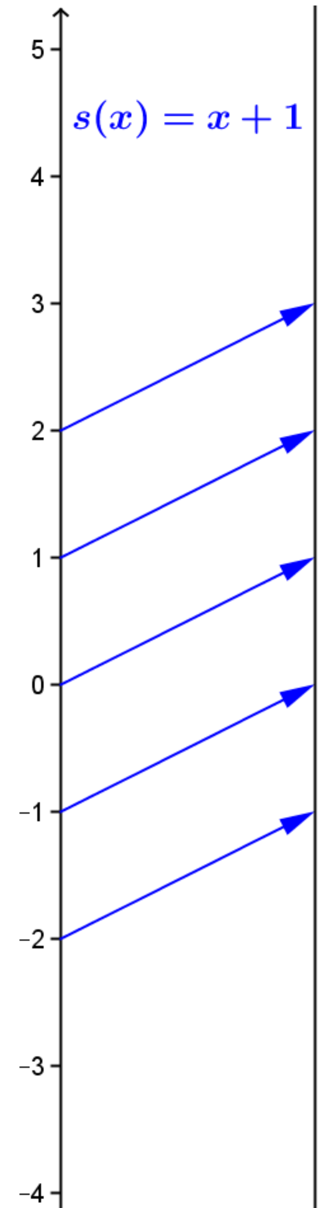
x	m(x) =2x
2	4
1	2
0	0
-1	-2
-2	-4



Worksheet 1.b Mapping Diagram:

$s(x) = x+1$

x	s(x)=x+1
2	3
1	2
0	1
-1	0
-2	-1



Technology Examples

- Excel examples
- GeoGebra examples

Mapping Diagram Prelim

- Examples of mapping diagrams
  - Worksheet 2
  - a. First make table for $q(x) = x^2$.

x	$q(x) = x^2$
2	
1	
0	
-1	
-2	

Mapping Diagram Prelim

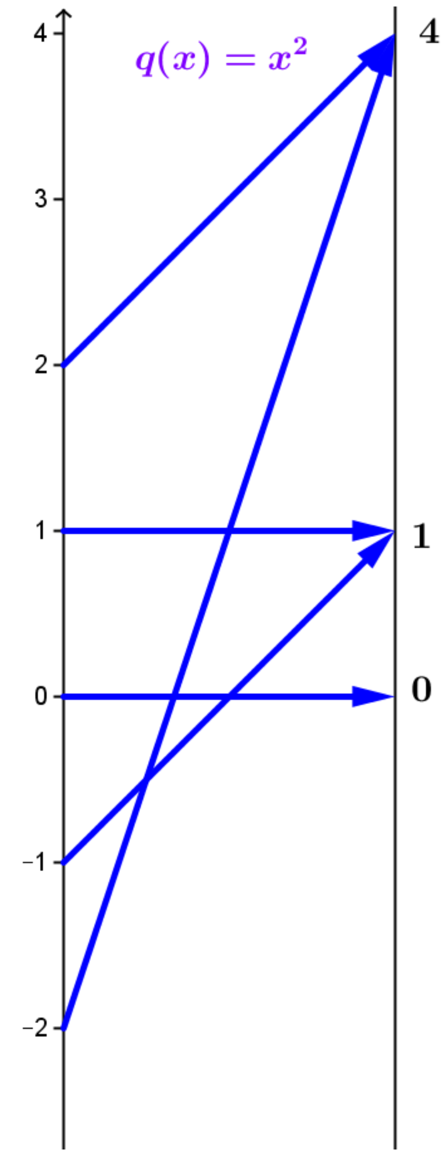
- Examples of mapping diagrams
  - Worksheet 2
  - a. First make table for q.

x	$q(x) = x^2$
2	4
1	1
0	0
-1	1
-2	4

  - b. Sketch a mapping diagram for $q(x) = x^2$.

Mapping Diagram Prelim
Worksheet 2.b. Mapping Diagram for $q(x) = x^2$

x	$q(x) = x^2$
2	4
1	1
0	0
-1	1
-2	4



Worksheet 3.a.Complete the following table for the composite function $f(x) = s(m(x)) = 2x + 1$

x	m(x)	f(x)=s(m(x))
2		
1		
0		
-1		
-2		

Worksheet 3.a.Complete the following table for the composite function $f(x) = s(m(x)) = 2x + 1$

x	m(x)	f(x)=s(m(x))
2	4	5
1	2	3
0	0	1
-1	-2	-1
-2	-4	-3

Mapping Diagram Prelim

- Worksheet 3.b
- Use the table 3.a and the previous sketches of 1.b to draw a composite sketch of the mapping diagram with <u>3 axes for the composite function</u>

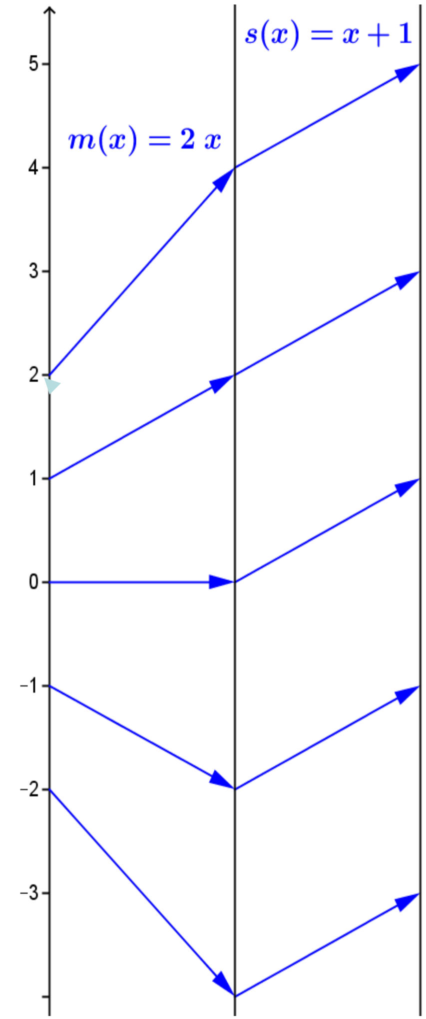
$$\underline{f(x) = h(g(x)) = 2x + 1}$$

Worksheet 3.b Draw a sketch for the mapping diagram with 3 axes of $f(x) = 2x + 1$.

x	m(x)	f(x)=s(m(x))
2	4	5
1	2	3
0	0	1
-1	-2	-1
-2	-4	-3

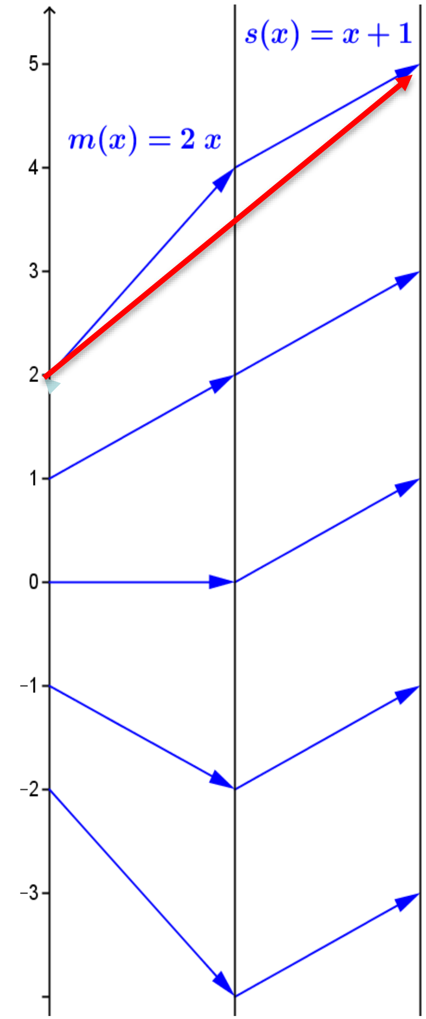
Worksheet 3.b Draw a sketch for the mapping diagram with 3 axes of $f(x) = 2x + 1$.

x	m(x)	f(x)=s(m(x))
2	4	5
1	2	3
0	0	1
-1	-2	-1
-2	-4	-3



Worksheet 3.c Draw a sketch for the mapping diagram with 2 axes of $f(x) = 2x + 1$.

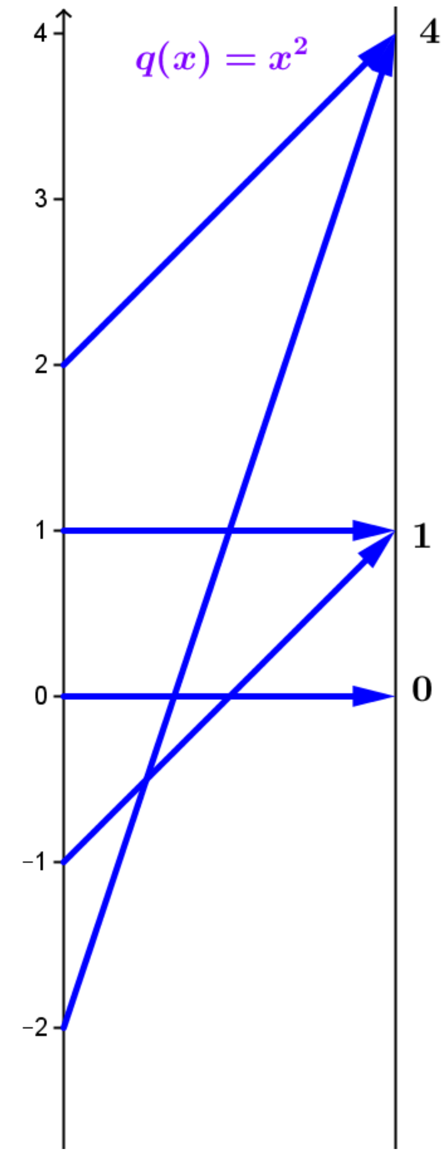
x	m(x)	f(x)=s(m(x))
2	4	5
1	2	3
0	0	1
-1	-2	-1
-2	-4	-3



Worksheet 4 Mapping Diagram:

$q(x) = x^2$

x	$q(x) = x^2$
2	4
1	1
0	0
-1	1
-2	4



Worksheet 4.a

Complete the following tables for $q(x) = x^2$ and $R(x) = s(q(x)) = x^2 + 1$

x	q(x)	R(x)=s(q(x))
2		
1		
0		
-1		
-2		

Worksheet 4.a

Complete the following tables for $q(x) = x^2$ and $R(x) = s(q(x)) = x^2 + 1$

x	q(x)	R(x)=s(q(x))
2	4	5
1	1	2
0	0	1
-1	1	2
-2	4	5

Worksheet 4.b

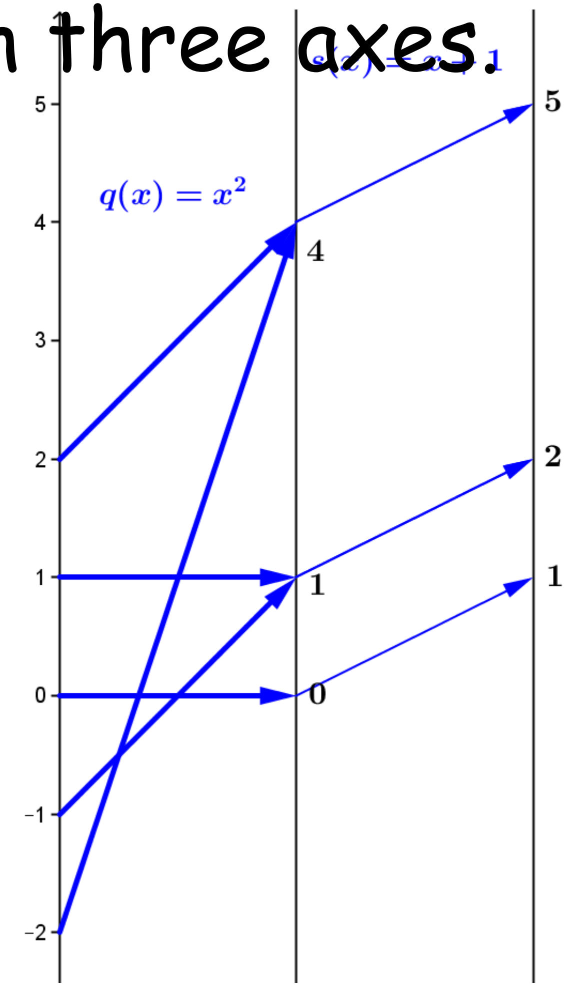
- 4.b Using the data from part a), sketch mapping diagrams for the composition $R(x) = s(q(x)) = x^2 + 1$ with three axes.

x	q(x)	R(x)=s(q(x))
2	4	5
1	1	2
0	0	1
-1	1	2
-2	4	5

Worksheet 4.b

- 4.b Using the data from part a), sketch mapping diagrams for the composition $R(x) = s(q(x)) = x^2 + 1$ with three axes.

x	q(x)	R(x)=s(q(x))
2	4	5
1	1	2
0	0	1
-1	1	2
-2	4	5



Worksheet 4.b

- 4.b Using the data from part a), sketch mapping diagrams for the composition $R(x) = s(q(x)) = x^2 + 1$ with two axes.

x	q(x)	R(x)=s(q(x))
2	4	5
1	1	2
0	0	1
-1	1	2
-2	4	5

Technology Examples

- Excel examples
- Geogebra examples

An Old Friend: Solving A Linear Equation

- Worksheet 5.a Solve a linear equation:

$$2x + 1 = 5$$

Find x.

An Old Friend:
Solving A Linear Equation

Worksheet 5.a Solve a linear equation:

$$2x + 1 = 5$$

$$-1 = -1$$

$$2x = 4$$

An Old Friend: Solving A Linear Equation

Worksheet 5.a Solve a linear equation:

$$2x + 1 = 5$$

$$-1 = -1$$

$$2x = 4$$

$$1/2(2x) = 1/2(4)$$

$$x = 2$$

An Old Friend: Solving A Linear Equation

Worksheet 5.a Solve a linear equation:

$$2x + 1 = 5$$

$$\underline{\qquad -1 = -1}$$

$$2x = 4$$

$$\underline{1/2(2x) = 1/2(4)}$$

$$x = 2$$

Check!

$$2x+1 = 2*2 + 1 \overset{!}{=} 5$$

Linear Equations Use Linear Functions!

Linear Equations

$2x + 1 = 5$

$-1 = -1$

$2x = 4$

$1/2(2x) = 1/2(4)$

$x = 2$

Check:

$2x + 1 = 2 * 2 + 1 = 5$

Linear Functions

$f(x) = 2x + 1$

Linear Equations Use Linear Functions!

Linear Equations

$2x + 1 = 5$

$-1 = -1$

$2x = 4$

$1/2(2x) = 1/2(4)$

$x = 2$

Check:

$2x + 1 = 2*2 + 1 = 5$

Linear Functions

$f(x) = 2x + 1$

$m(x) = 2x; s(x) = x + 1$

$f(x) = s(m(x))$

Worksheet 5.b Solving 2x + 1 = 5 visualized with a mapping diagram

Algebra:

2x + 1 = 5

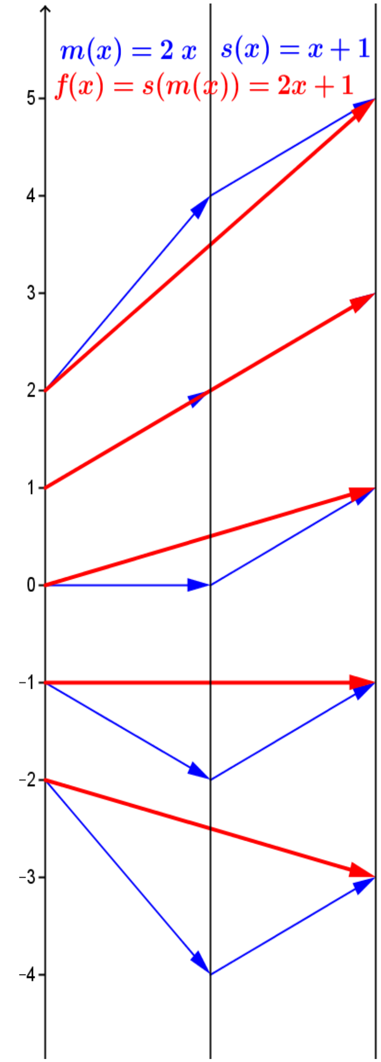
-1 = -1

2x = 4

1/2(2x) =1/2(4)

x = 2

How does the MD for the function VISUALIZE the algebra?



Worksheet 5.b Solving 2x + 1 = 5 visualized with a mapping diagram

Algebra:

$2x + 1 = 5$

$-1 = -1$

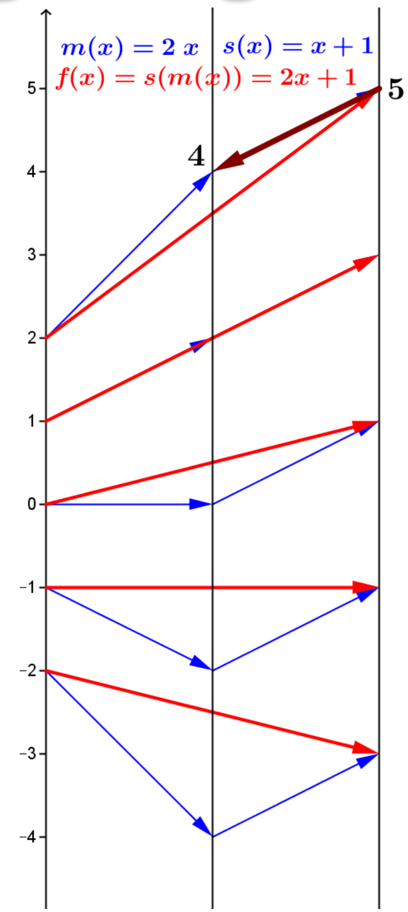
$2x = 4$

Function:

$f(x)=s(m(x)) = 5$

"Undo s"

$m(x) = 4$



Worksheet 5.b Solving 2x + 1 = 5 visualized with a mapping diagram

Algebra:

$2x + 1 = 5$

$-1 = -1$

$2x = 4$

$1/2(2x) = 1/2(4)$

$x = 2$

Function:

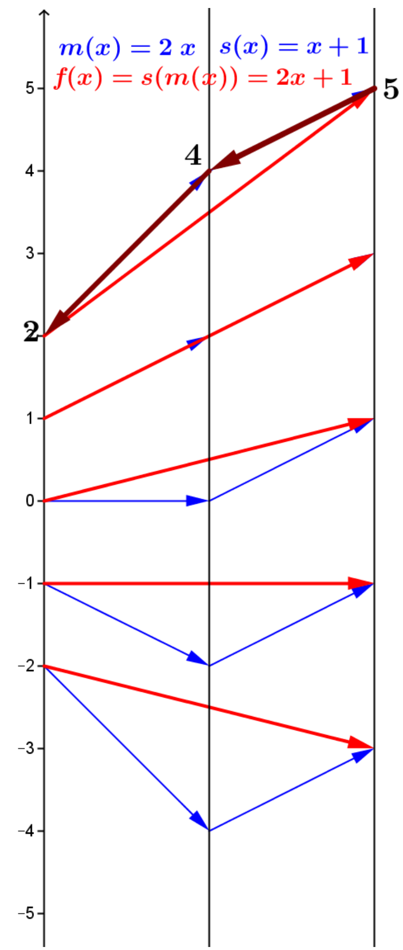
$f(x) = s(m(x)) = 5$

"Undo s"

$m(x) = 4$

"Undo m"

$x = 2$



Worksheet 5.b Solving 2x + 1 = 5 visualized with a mapping diagram

Algebra:

$2x + 1 = 5$

$-1 = -1$

$2x = 4$

$1/2(2x) = 1/2(4)$

$x = 2$

Function:

$f(x)=s(m(x)) = 5$

"Undo s"

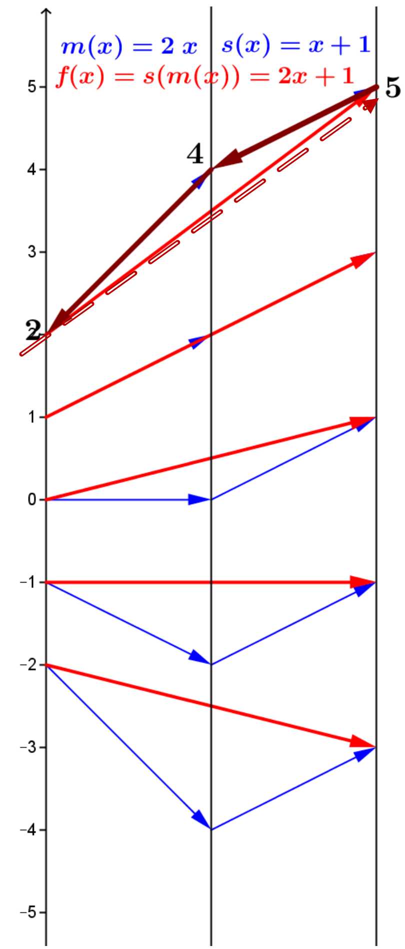
$m(x) = 4$

"Undo m"

$x = 2$

CHECK! ☺

$f(2)\overset{!}{=}5$



Worksheet 5.b Solving 2x + 1 = 5 visualized on GeoGebra

Algebra:

$2x + 1 = 5$

$-1 = -1$

$2x = 4$

$1/2(2x) = 1/2(4)$

$x = 2$

Function:

$f(x)=s(m(x)) = 5$

"Undo s"

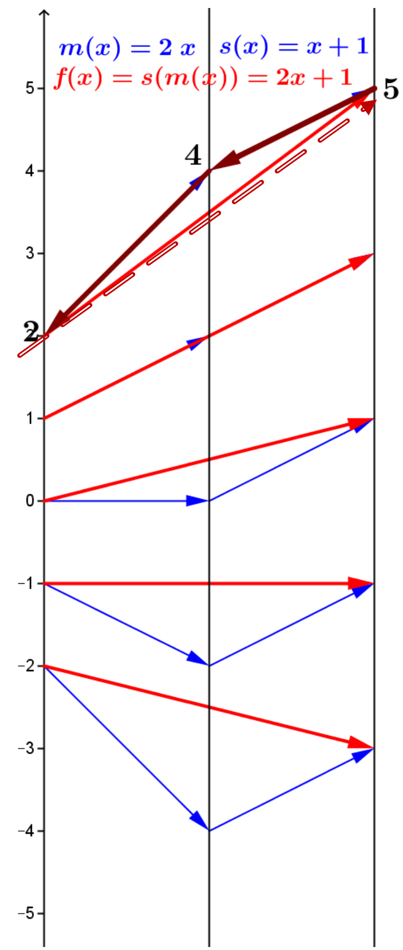
$m(x) = 4$

"Undo m"

$x = 2$

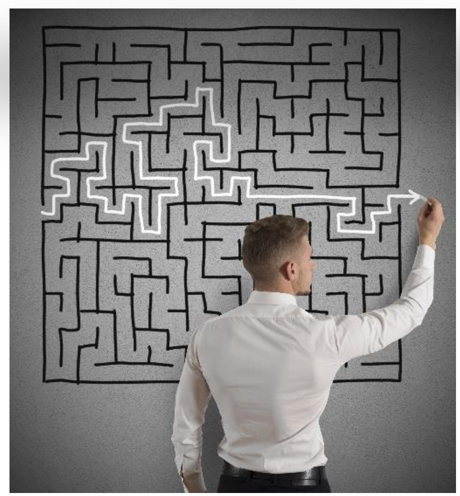
CHECK! ☺

$f(2) \overset{!}{=} 5$



Challenge: Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

The Great Mind Challenge
Initiate Collaborate Innovate



Worksheet 6.a Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Understand the problem

- $2(x-3)^2 + 1$ is a function of x.
  - $P(x) = 2(x-3)^2 + 1$
- Find any and all x where $P(x) = 9$.
- $2(x-3)^2 + 1$ is a composition of functions
  - $P(x) = s(m(q(z(x))))$ where
  - $z(x) =$
  - $q(x) =$
  - $m(x) =$
  - $s(x) =$

Worksheet 6.a Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Understand the problem

- $2(x-3)^2 + 1$ is a function of x.
  - $P(x) = 2(x-3)^2 + 1$
- Find any and all x where $P(x) = 9$.
- $2(x-3)^2 + 1$ is a composition of functions
  - $P(x) = s(m(q(z(x))))$ where
  - $z(x) = x-3$;
  - $q(x) = x^2$;
  - $m(x) = 2x$;
  - $s(x) = x+1$.

Worksheet 6.a Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram.

Make a plan

- Find any and all x where P(x) = 9.
- Construct mapping diagram for P as a composition of function :

$$P(x) = s(m(q(z(x))))$$

- Undo P(x) = 9 by undoing each step of P
  - Undo $s(x) = x+1$
  - Undo $m(x) = 2x$
  - Undo $q(x) = x^2$
  - Undo $z(x) = x-3$
- Check results to see that P(x) = 9

Worksheet 6.b Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram.

Execute the **plan**

– Construct mapping diagram for P as a composition of function :

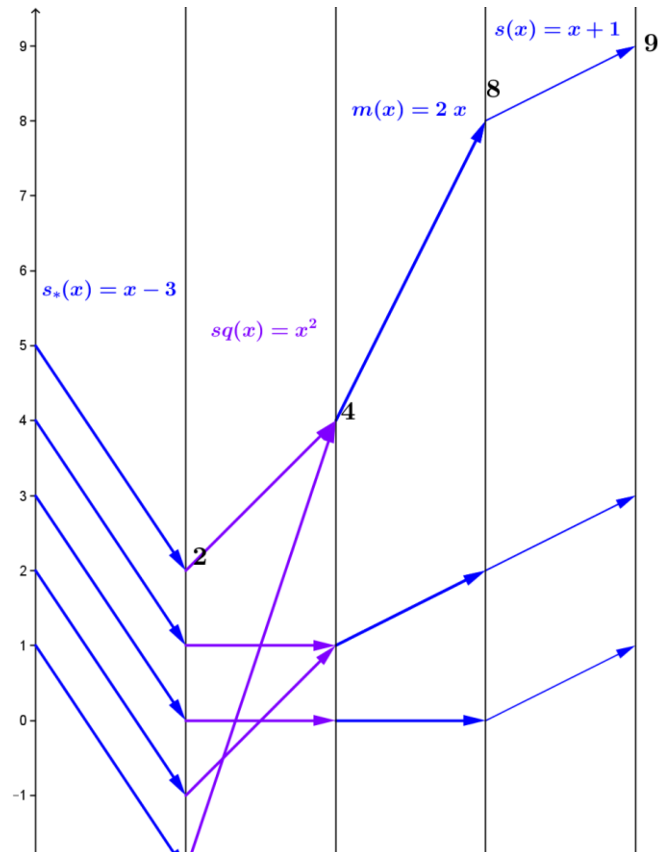
$$P(x) = s(m(q(z(x))))$$

Worksheet 6.b Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram.

Execute the **plan**

- Construct mapping diagram for P as a composition of function :

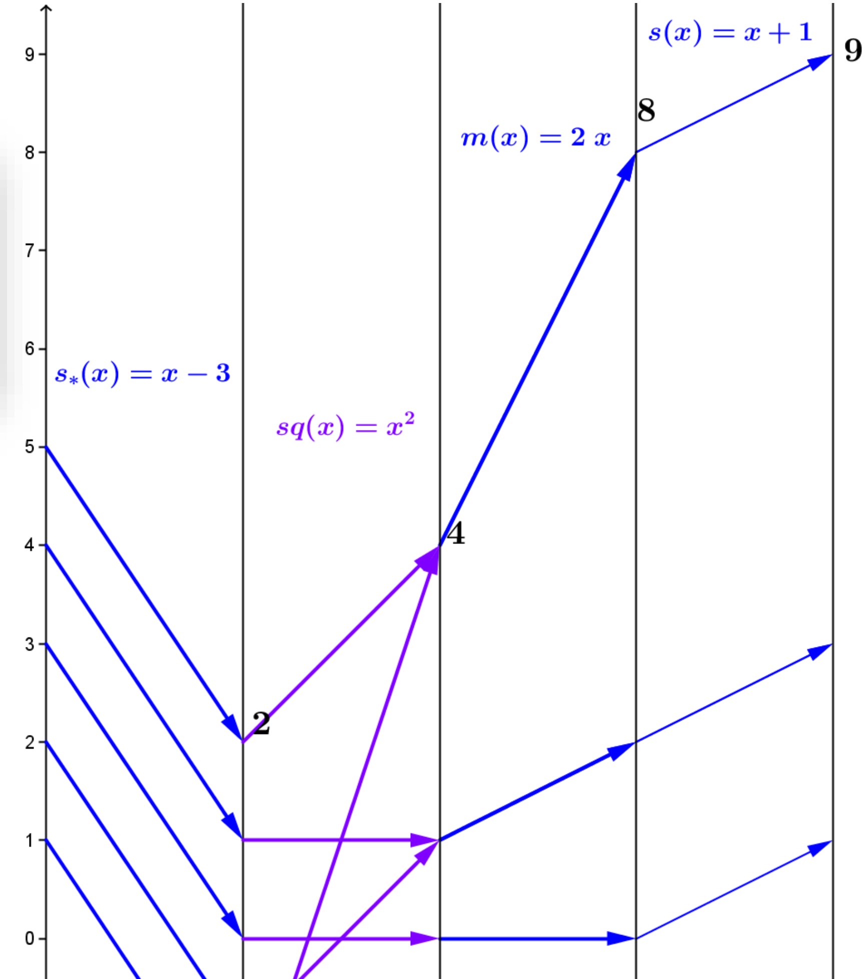
$$P(x) = s(m(q(z(x))))$$



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Execute the plan

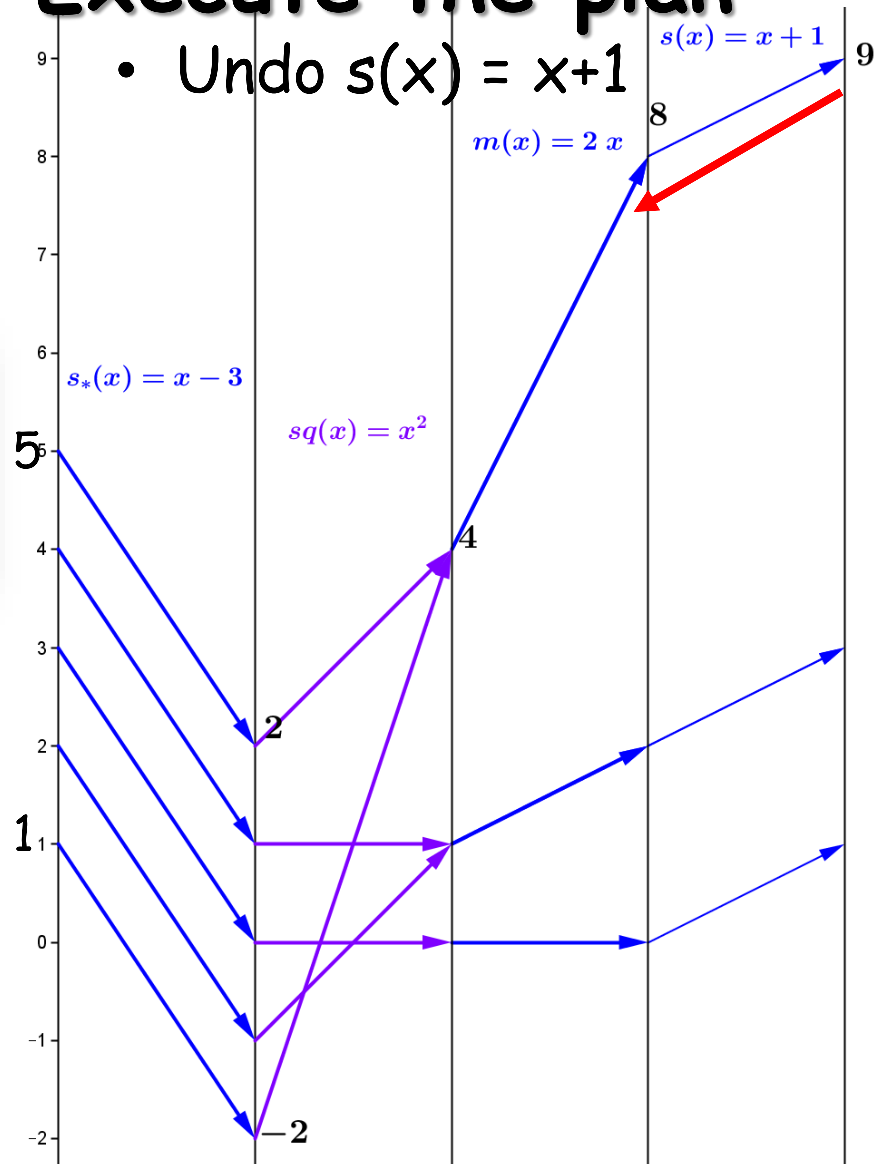
- Find any and all x where P(x) = 9.



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Execute the plan

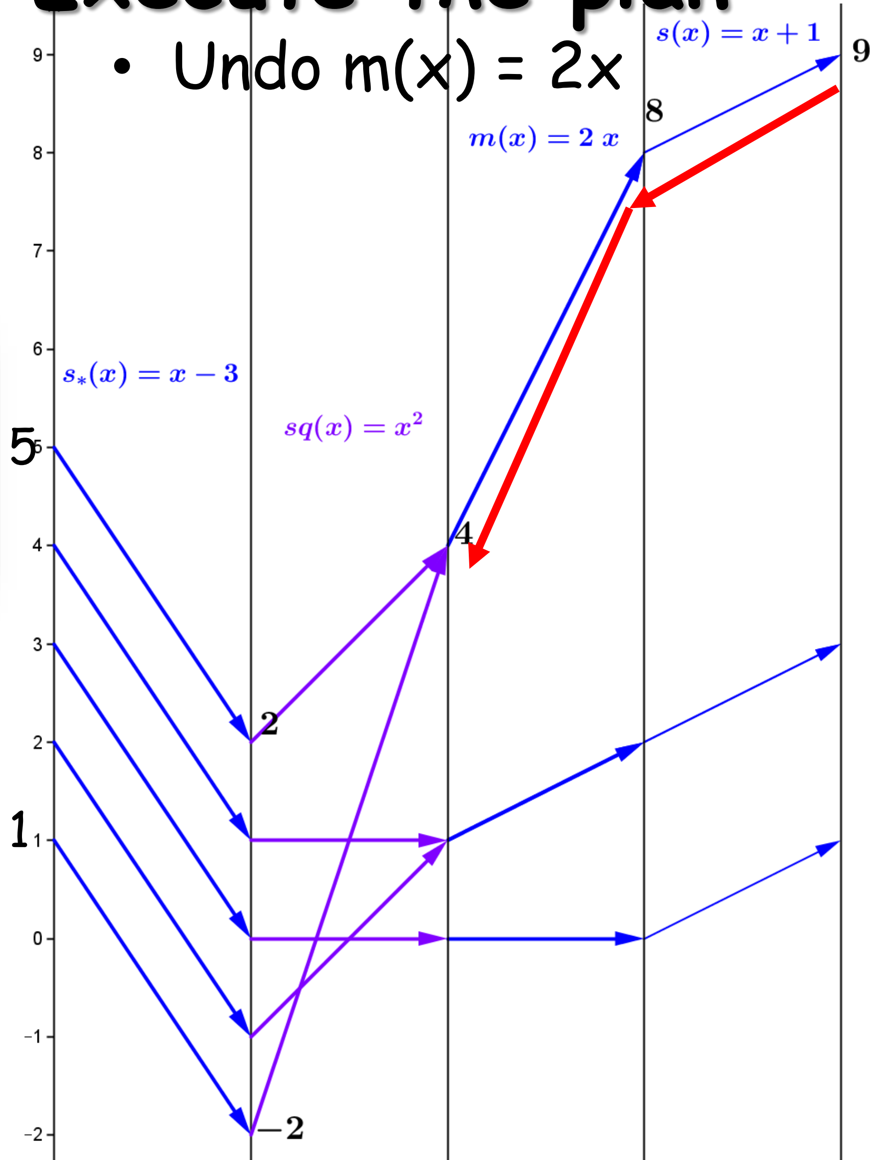
- Undo $s(x) = x+1$



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Execute the plan

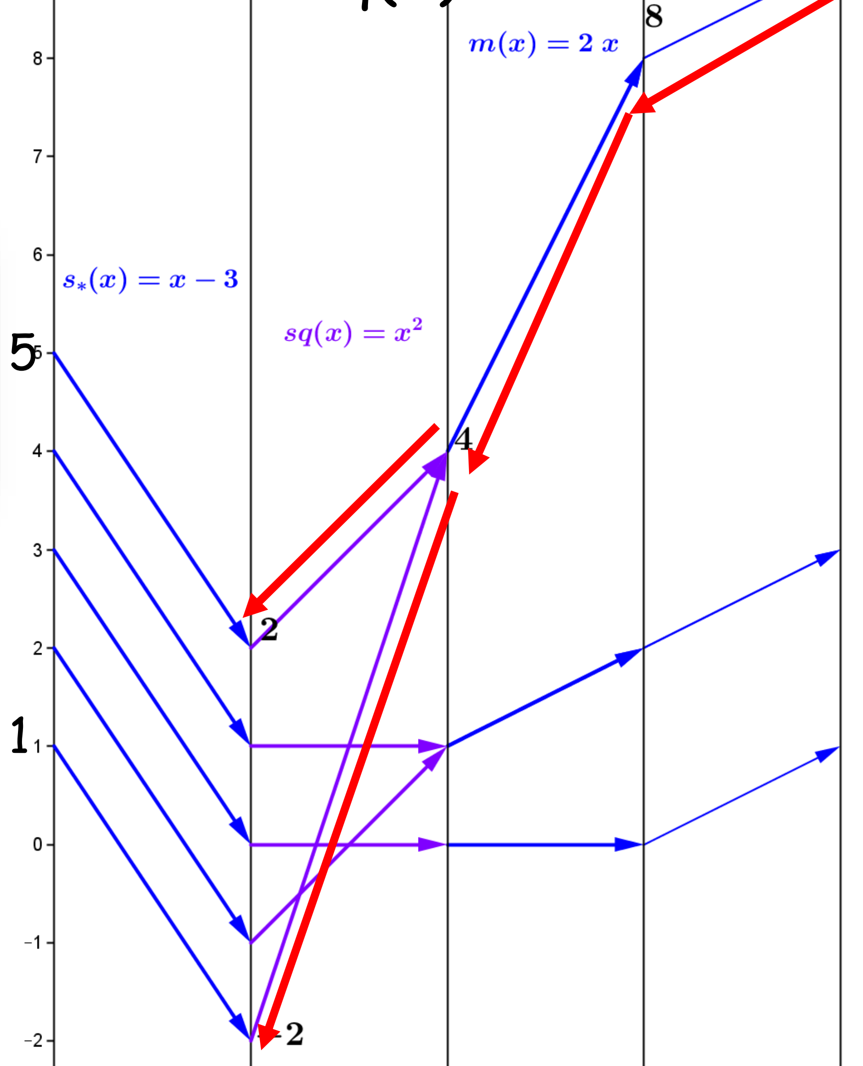
- Undo m(x) = 2x



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Execute the plan

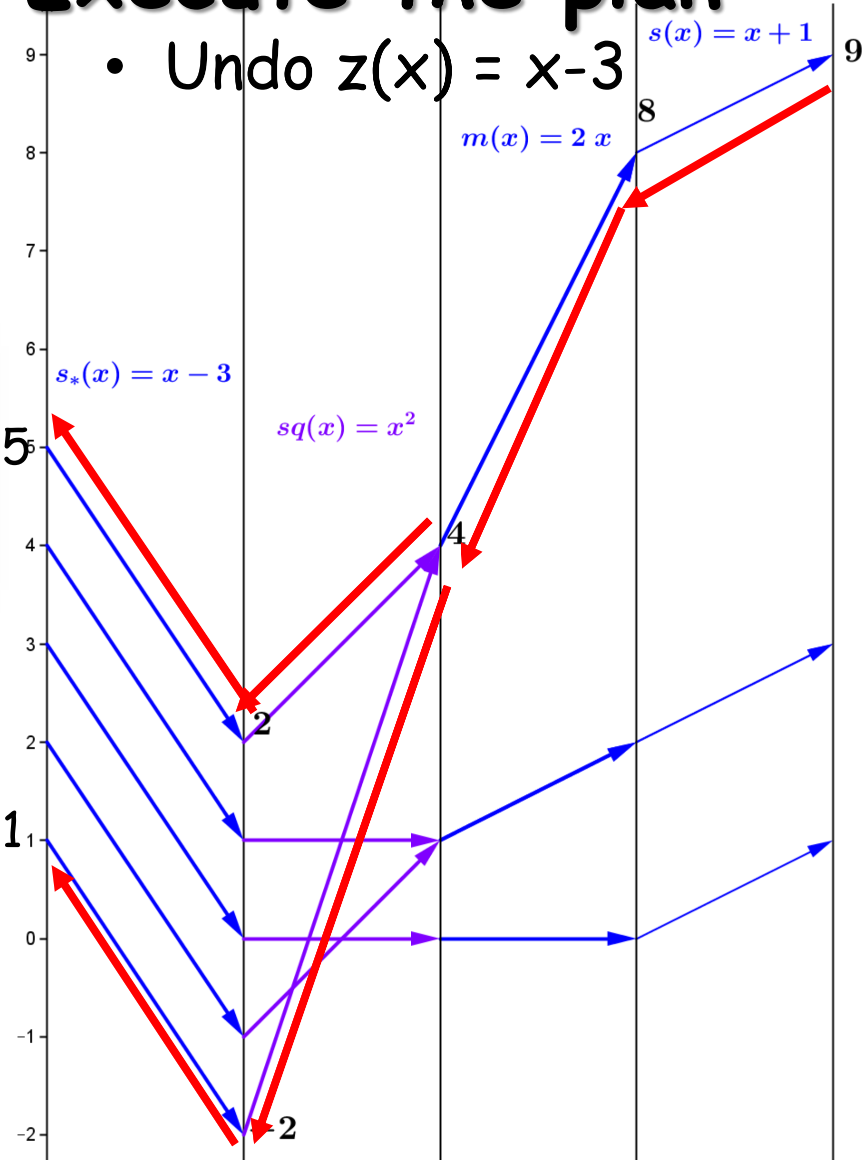
- Undo $q(x) = x^2$



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

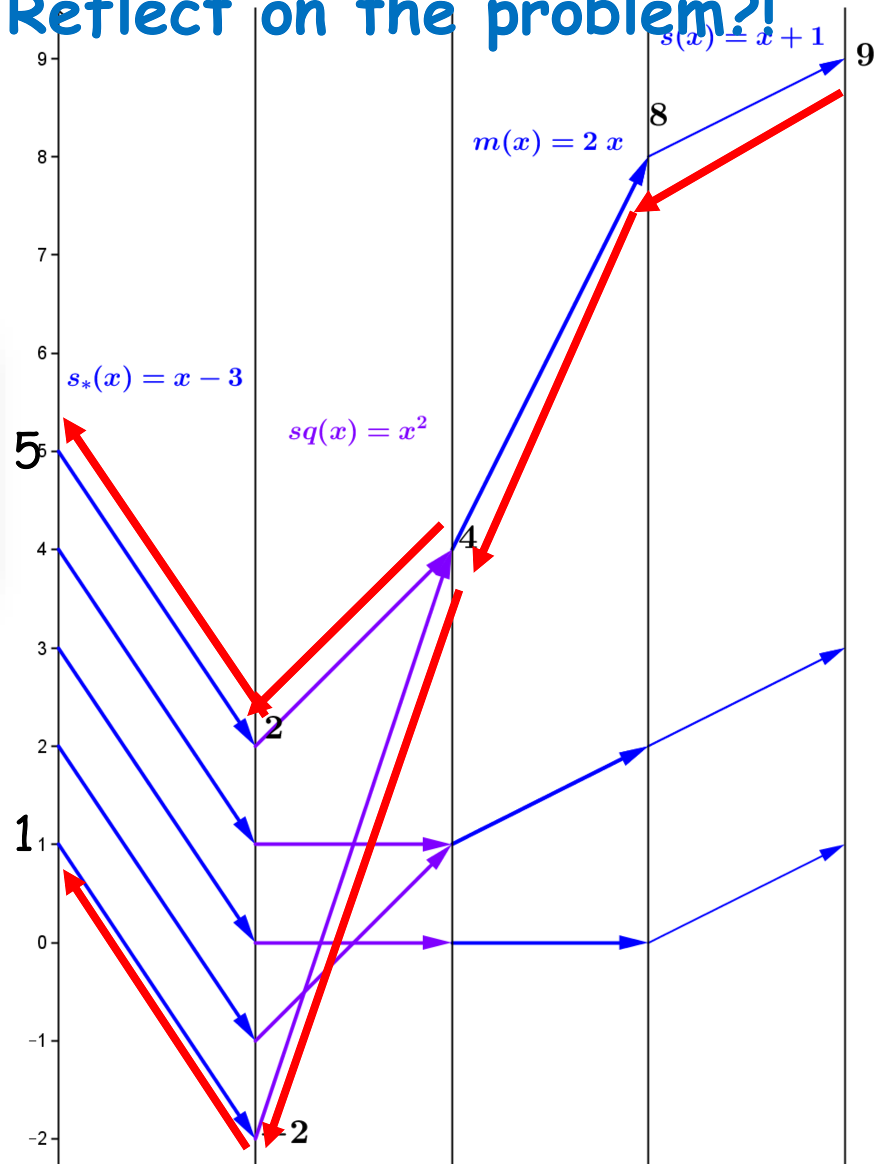
Execute the plan

- Undo z(x) = x-3



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Reflect on the problem?!

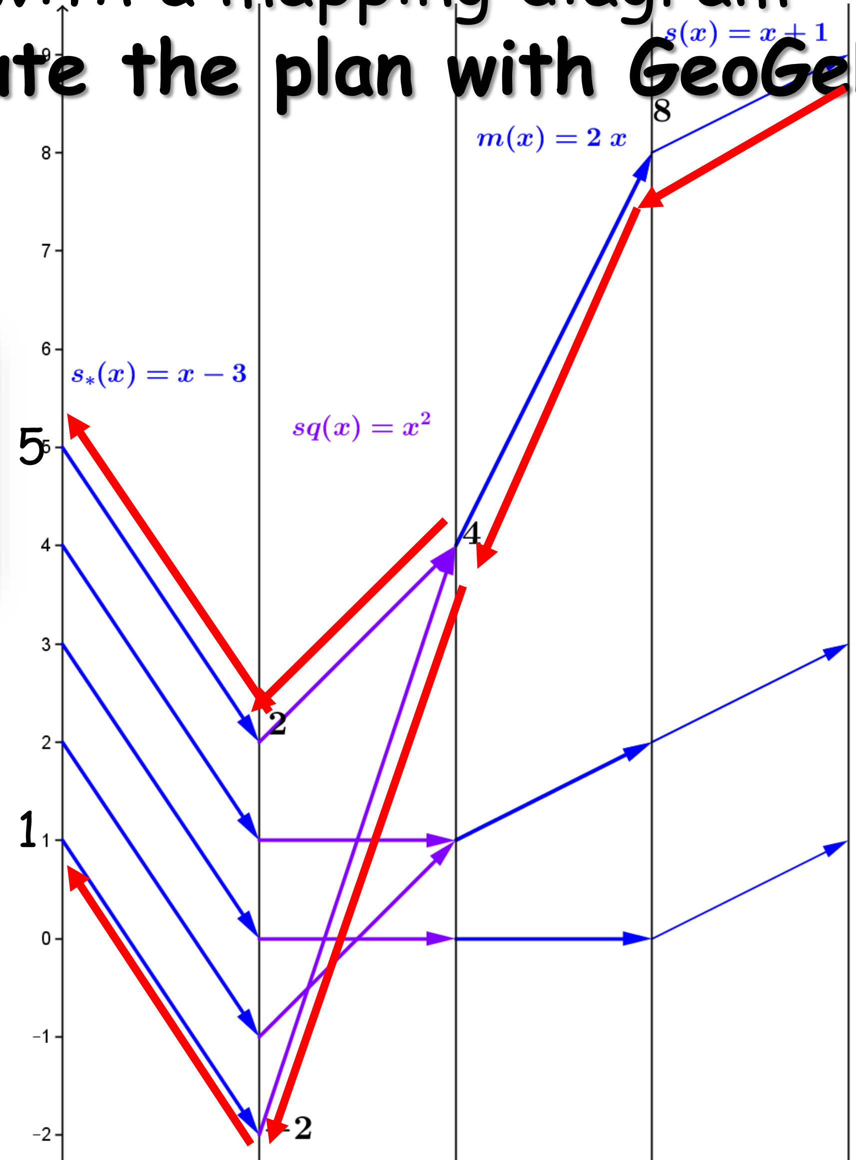


Challenge: Worksheet 6.c

Solve $2(x-3)^2 + 1 = 9$

with a mapping diagram

Execute the plan with GeoGebra



iii. A Look Beyond

- Complex Numbers
- The Complex Plane
- Complex Functions
- Mapping Diagrams in 3 Dimensions
- GeoGebra

Technology Examples

- Excel examples
- Geogebra examples

Overtime?
Simple Examples are important!

- $f(x) = x + C$ Added value: C
- $f(x) = mx$ Scalar Multiple: m

  Interpretations of m:
  - slope
  - rate
  - Magnification factor
  - $m > 0$: Increasing function
  - $m < 0$: Decreasing function
  - $m = 0$: Constant function

Simple Examples are important!

f(x) = mx + b with a mapping diagram --
Five examples:
Back to Worksheet Problem #7

- Example 1: m =-2; b = 1: $f(x) = -2x + 1$
- Example 2: m = 2; b = 1: $f(x) = 2x + 1$
- Example 3: m = $\frac{1}{2}$; b = 1: $f(x) = \frac{1}{2}x + 1$
- Example 4: m = 0; b = 1: $f(x) = 0x + 1$
- Example 5: m = 1; b = 1: $f(x) = x + 1$

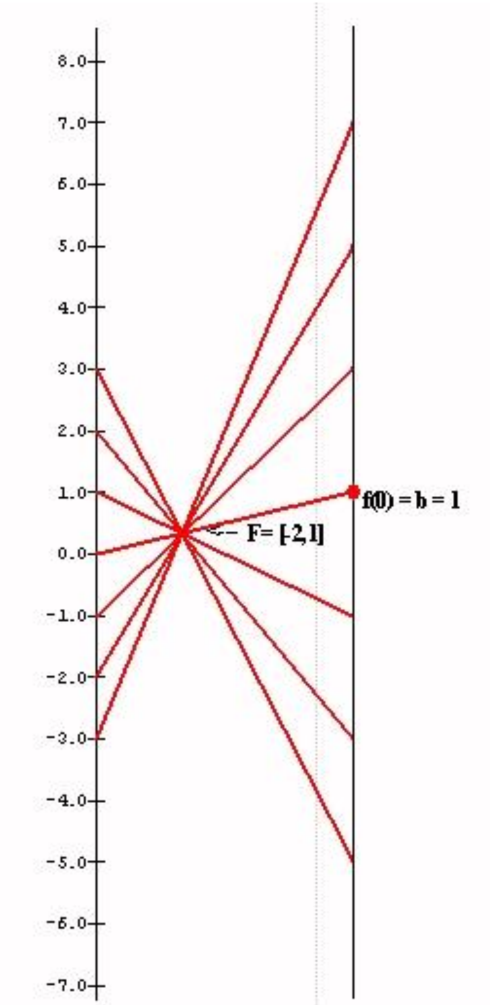
Visualizing $f(x) = mx + b$ with a mapping diagram -- Five examples:

Example 1: m = -2; b = 1

$f(x) = -2x + 1$

- Each arrow passes through a single point, which is labeled **F = [- 2,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, -2*x* + 1,

which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = \boldsymbol{m}x + \boldsymbol{b}$ with a mapping diagram -- Five examples:

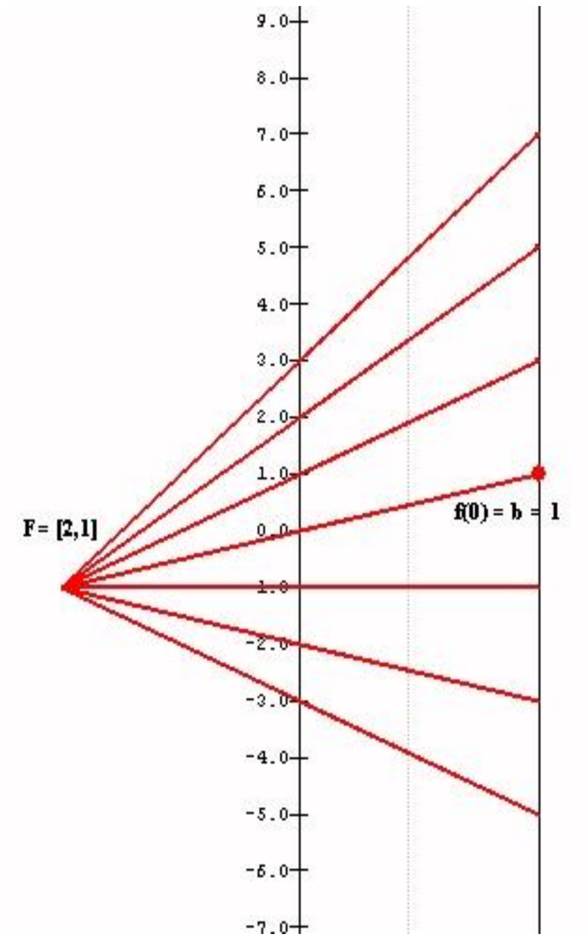
Example 2: *m* = 2; *b* = 1
$f(x) = 2x + 1$

Each arrow passes through a single point, which is labeled

F = [2,1].

- The point **F** completely determines the function *f*.
  - **given a point** / number, *x*, on the source line,
  - there is a **unique arrow passing through F**
  - **meeting** the target line at **a unique point** / number, $2x + 1$,

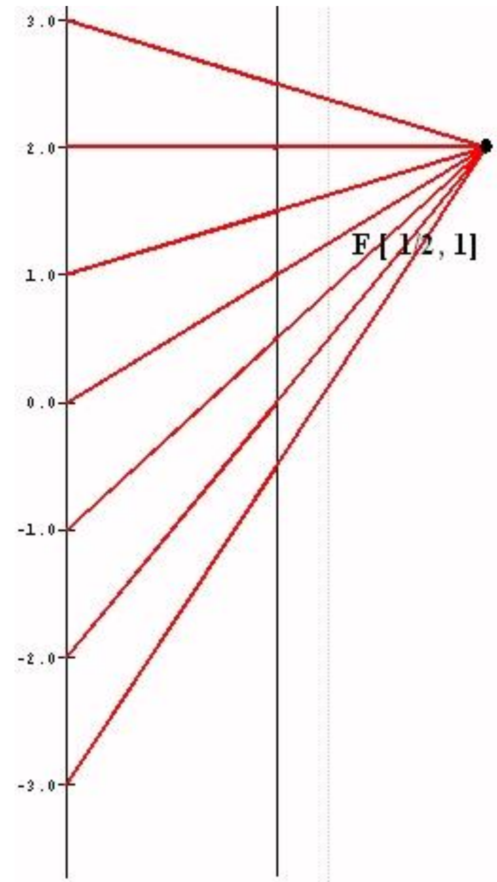
which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = mx + b$ with a mapping diagram -- Five examples:

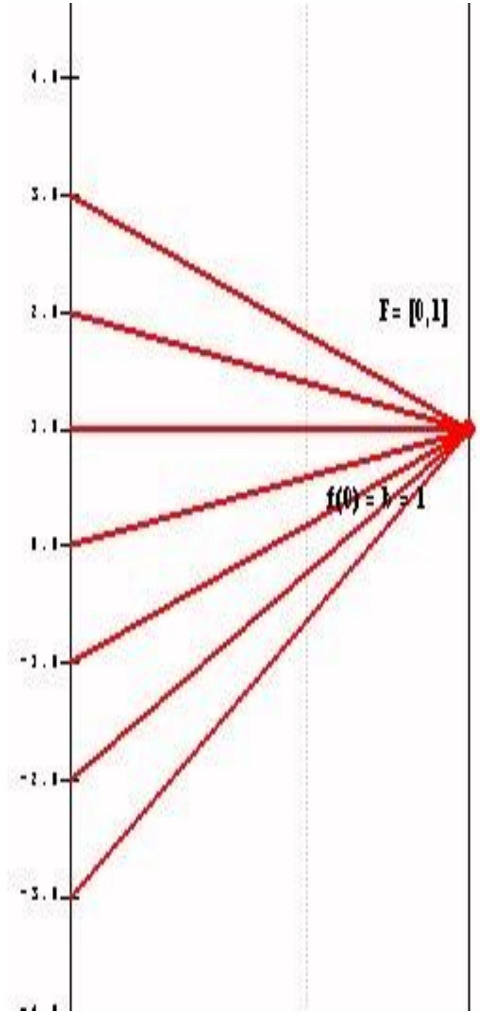
- **Example 3: $m = 1/2$; $b = 1$**
  $f(x) = \frac{1}{2} x + 1$
- Each arrow passes through a single point, which is labeled **F = [1/2,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $\frac{1}{2} x + 1$,

    which corresponds to the linear function's value for the point/number, *x*.



Visualizing *f* (*x*) = ***mx*** + ***b*** with a mapping diagram -- Five examples:

- **Example 4: *m* = 0; *b* = 1**
  f(x) = 0 x + 1



- Each arrow passes through a single point, which is labeled **F = [0,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, *f*(*x*)=1,

    which corresponds to the linear function's value for the point/number, *x*.

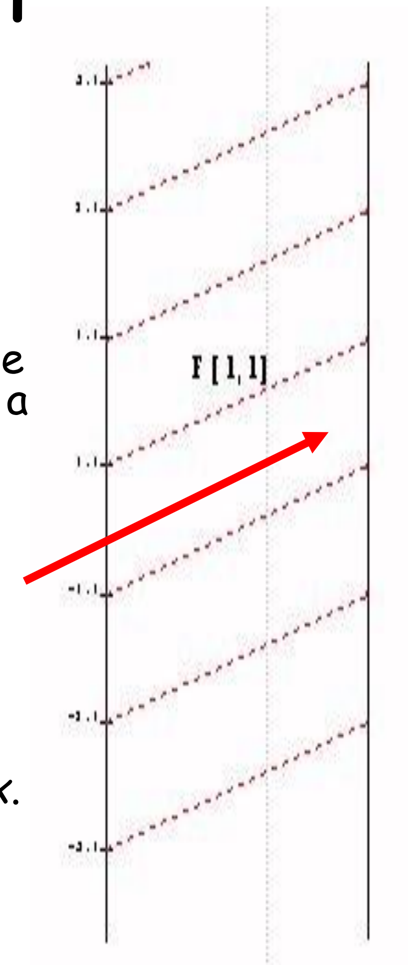
Visualizing $f(x) = \boldsymbol{mx} + \boldsymbol{b}$ with a mapping diagram -- Five examples

Example 5: $m = 1$; $b = 1$

$f(x) = x + 1$

- Unlike the previous examples, in this case it is not a single point that determines the mapping diagram, but the single arrow from 0 to 1, which we designate as F[1,1]
- It can also be shown that this single arrow completely determines the function.Thus, given a point / number, *x*, on the source line, there is a unique arrow passing through ***x parallel to*** F[1,1] meeting the target line a unique point / number, $x + 1$, which corresponds to the linear function's value for the point/number, *x*.
    - The single arrow completely determines the function *f*.
        - **given a point** / number, *x*, on the source line,
        - there is a **unique arrow** through ***x parallel to*** F[1,1]
        - **meeting** the target line at **a unique point** / number, $x + 1$,

which corresponds to the linear function's value for the point/number, *x*.



Simple Examples are important!

- $f(x) = x + C$ Added value: C
- $f(x) = mx$ Scalar Multiple: m

  Interpretations of m:
  - slope
  - rate
  - Magnification factor
  - $m > 0$: Increasing function
  - $m < 0$: Decreasing function
  - $m = 0$: Constant function

Function-Equation Questions

with linear focus points (Problem 8.a)

- Use a focus point in the mapping diagram to solve a linear equation:

$$2x+1 = 5$$

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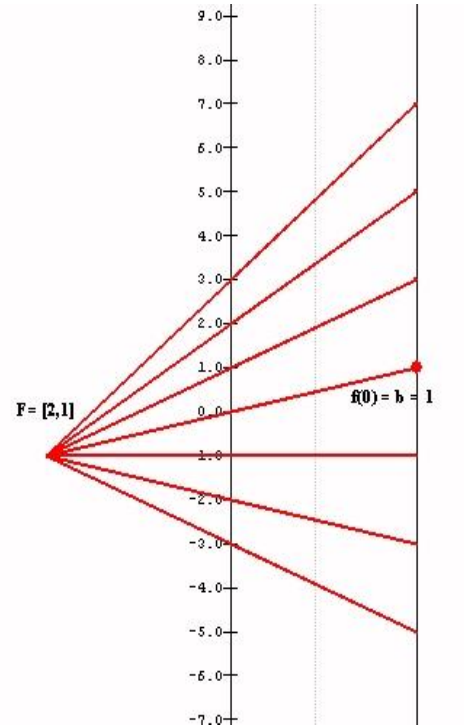

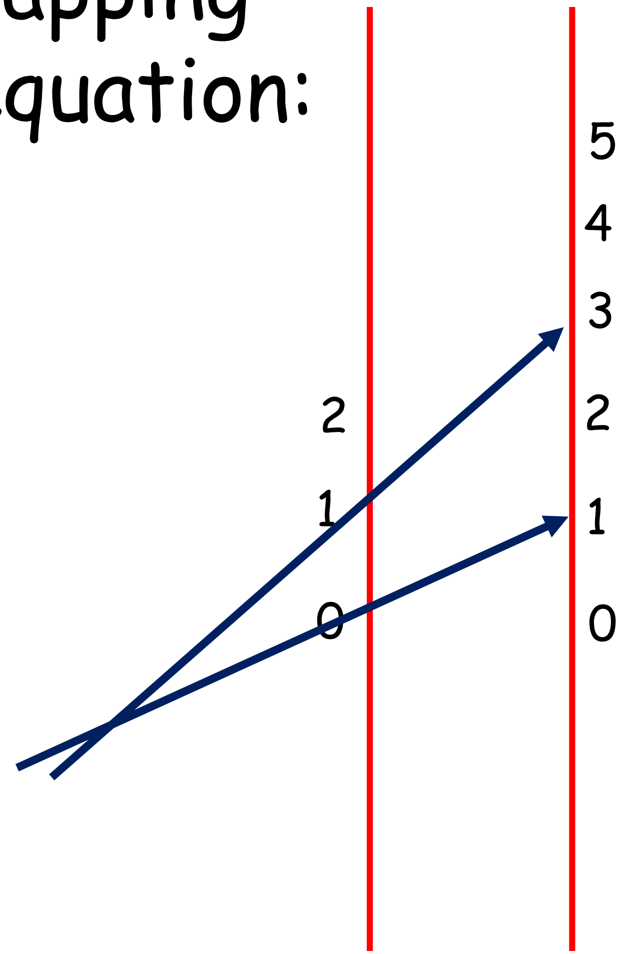

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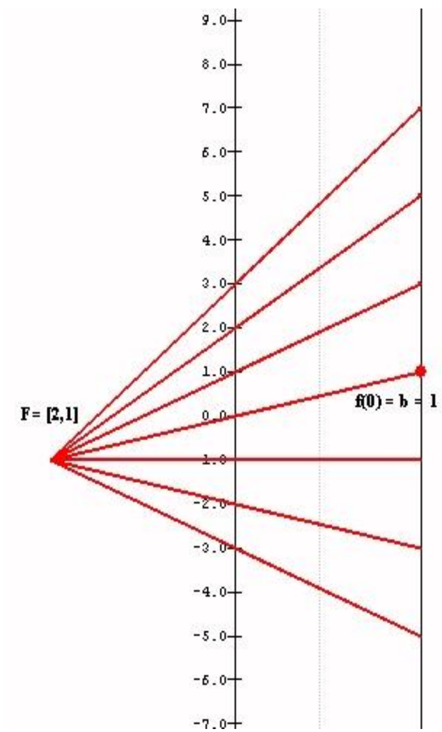

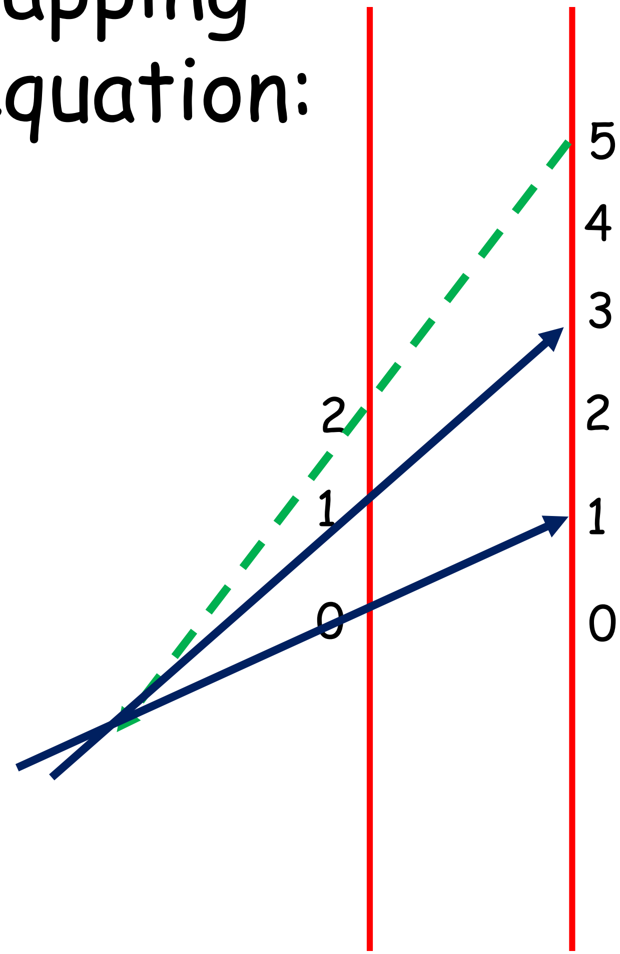

Function-Equation Questions

with linear focus points (Problem 8)

Suppose f is a linear function with f (1) = 3 and f (3) = -1.

Without algebra

- 8.b Use a focus point to find f (0).
- 8.c Use a focus point to find x where f (x) = 0.

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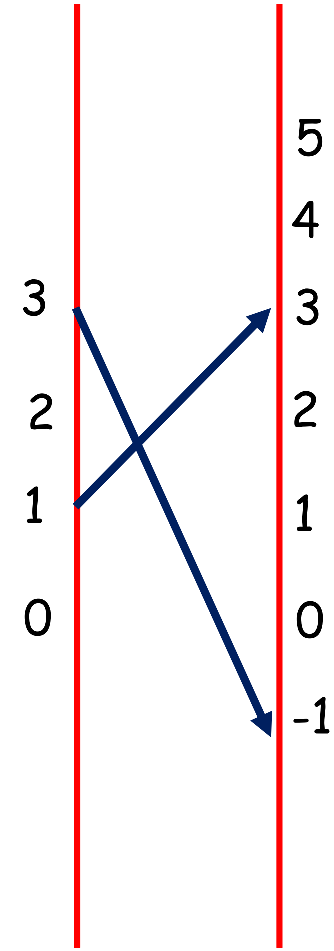

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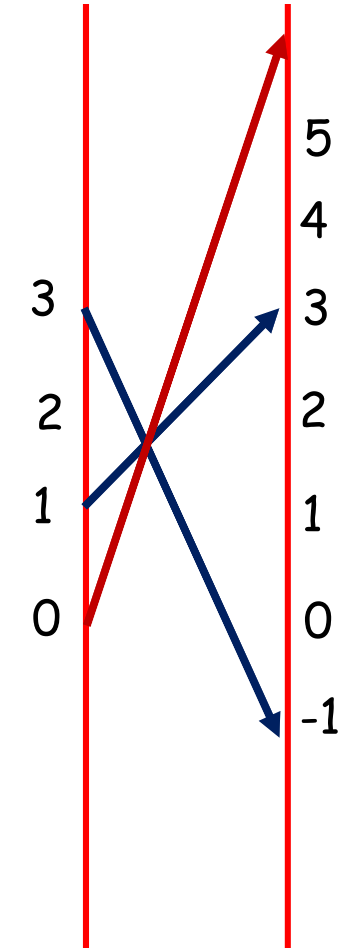

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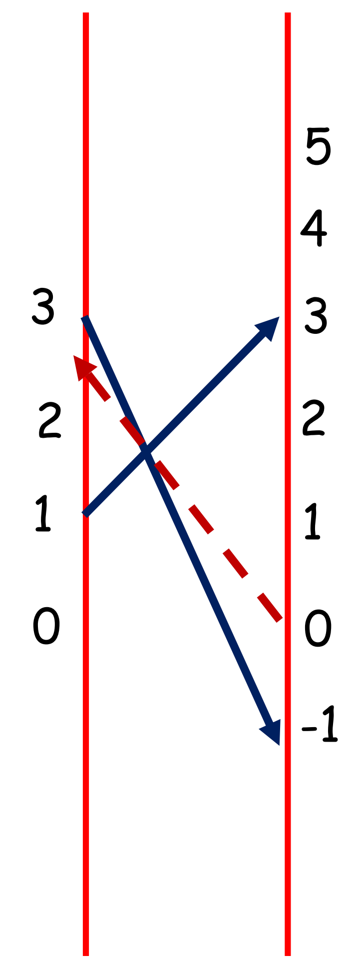

Thanks
The End!

☺

Questions?

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References

- Solving Linear Equations Visualized with Mapping Diagrams (YouTube) by M. Flashman

- **Function Diagrams. by Henri Picciotto Excellent Resources!**
  - **Henri Picciotto's Math Education Page**
  - **Some rights reserved**

- Mapping Diagrams from A(lgebra) B(asics) to C(alculus) and D(ifferential) E(quation)s. A Reference and Resource Book on Function Visualizations Using Mapping Diagrams (Preliminary Sections- NOT YET FOR publication) **http://users.humboldt.edu/flashman/MD/section-1.1VF.html**

- **Mapping Diagrams and Graphs…** Visualizing linear functions using mapping diagrams and graphs. **tube.geogebra.org** Martin Flashman

Thanks

The End!   REALLY!

☺

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