A Visual Validation of Viéte's Verification

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Nearly every college student, at some point in his or her education, encounters the quadratic equation, which gives the solutions to the equation $ax^2 + bx + c = 0$.

Of course, there are similar formulas for the roots of cubic and quartic functions, but these formulas are rather complicated and are usually absent from high school and college curricula. However, the solution to cubic equations can be understood by college mathematics students as they learn about different types of transformations of functions since the solution to the cubic follows from a series of transformations. We wondered what these transformations do to the graphs of the associated functions; consequently, in this article, we visually investigate the necessary transformations to solve the cubic using the technique of Viète (which was derived independently of the work of Tartaglia/Cardano).

A generic cubic equation has the form $x^3 + bx^2 + cx + d = 0$ where b, c, and d are real numbers (if the coefficient of x^3 is not 1, then we can divide the equation by that coefficient). By the fundamental theorem of algebra, this equation has three solutions (counting repetition). We perform a preliminary transformation (replacing x by $x - b/3$) that shifts the graph horizontally, moving the sum of the roots to 0. This transformation reduces the cubic to the so-called depressed cubic $x^3 + px + q = 0$. See [**1, 2, 3, 6**] for more detailed or historical accounts about solving the cubic.

Here, we provide a visual tour of the solution of the depressed cubic.

Transforming the cubic

Figure 1 depicts a depressed cubic along with the following three transformations (moving clockwise from the upper left graph).

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1. $x = \varphi(w) = w - p/(3w)$ (note that this is a two-to-one transformation as $\varphi(w) = \varphi(-p/(3w))$ for all w),
2. $z = w^3 y$,
3. $t = \theta(w) = w^3$.

The red (gray) dot in each of these graphs corresponds to the real root of the depressed cubic. We could have chosen either root in the upper right graph (or the lower graphs) to label by the dot. The result of the transformations is the function

$$z = t^2 + qt - \left(\frac{p}{3}\right)^3$$

whose roots can be determined using the quadratic equation.

We invite the reader to use a pen and paper to verify the accuracy of the equations and the corresponding coordinates of the red dot given in Figure 1. We recommend starting in the upper left and working clockwise to verify the formulas for the functions then starting in the lower left and moving counterclockwise to trace the solutions back from t to w to x giving a root of $y = x^3 + px + q$. It may be helpful to note that

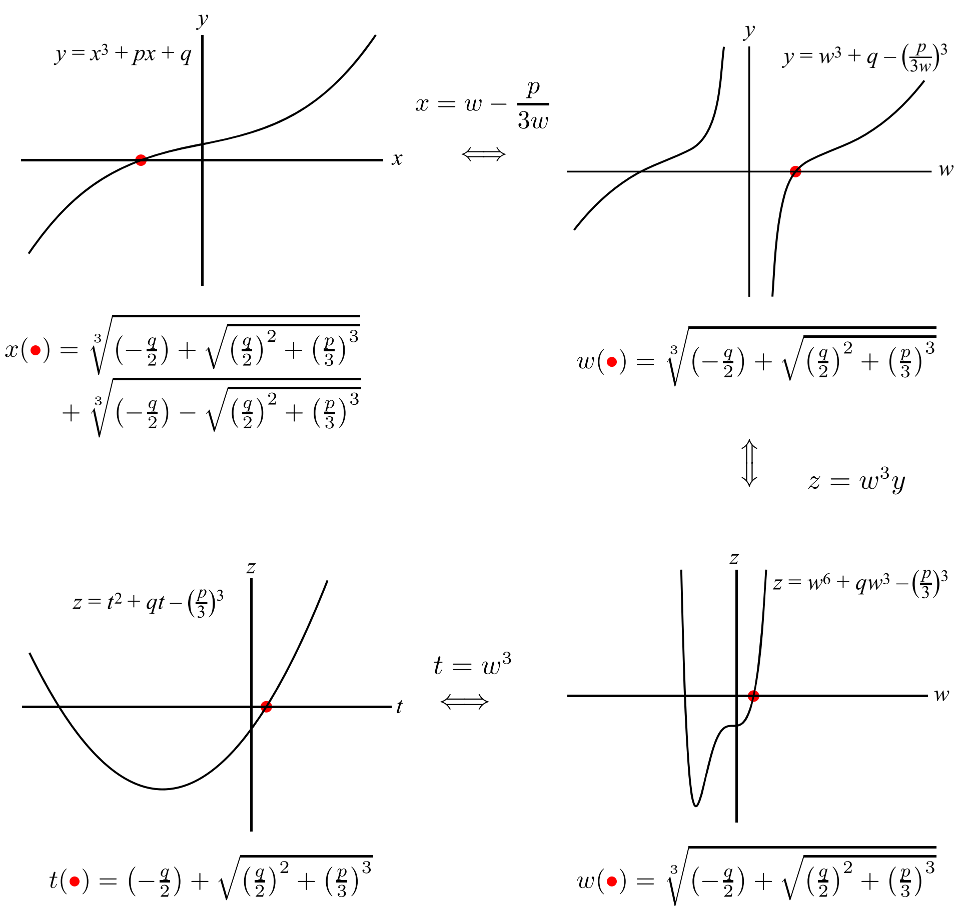

Figure 1. Visual solution of a depressed cubic using Viète's method when the discriminant $\Delta < 0$.

$$\frac{1}{\sqrt[3]{u+\sqrt{v}}} = \frac{\sqrt[3]{u-\sqrt{v}}}{\sqrt[3]{u^2-v}}$$

when both sides are defined.

The visual roadmap given in Figure 1 provides evidence for the following well-known theorem, and, in fact, the theorem can be proven algebraically using the three described transformations.

Theorem 1. *Let p and q be integers. One of the solutions of the depressed cubic equation $x^3 + px + q = 0$ is of the form*

$$a = \sqrt[3]{\left(-\frac{q}{2}\right) + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\left(-\frac{q}{2}\right) - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

While Figure 1 clearly expresses the proof concept of Theorem 1, it is only a proof sketch since there are three categories of depressed cubics, depending on the discriminant $\Delta = -(q/2)^2 - (p/3)^3$ of the cubic. If $\Delta < 0$ (as in Figure 1), then the cubic has one real root and two (conjugate) nonreal roots. If $\Delta > 0$, then the cubic has three distinct real roots. Finally, if $\Delta = 0$, then the cubic has a one real simple root and one real double root (unless the cubic has one real triple root, in which case the depressed cubic is simply $y = x^3$). We visualize a proof of Theorem 1 in each of these three cases separately below.

The graphs of the real plots in Figure 1 do not allow us to visualize the two complex roots. In order to do this, we will use the `complex_plot` command in Sage [**4**] to visualize complex functions. According to the Sage documentation [**5**], the resulting image shows the domain of the function and colors the plane so that "The magnitude of the output is indicated by the brightness (with zero being black and infinity being white) while the argument is represented by the hue (with red being positive real, and increasing through orange, yellow, ... as the argument increases)."

For instance, Figure 2 shows the complex plot of the identity function $y = x$.

Negative discriminant. Figure 3 shows plots of the complex domain of each of the four functions listed in Figure 1; we can see the complex roots of each function as points where all of the colors meet.

The plot in the upper left shows the real root of equation $y = x^3 + px + q$ as well as the two complex roots (which are conjugate). Similarly, the upper right plot shows six

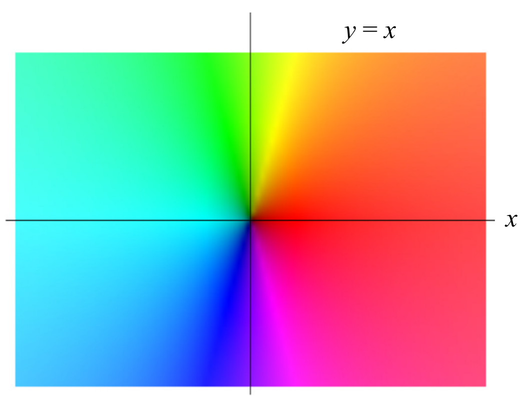

Figure 2. Complex plot for $y = x$.

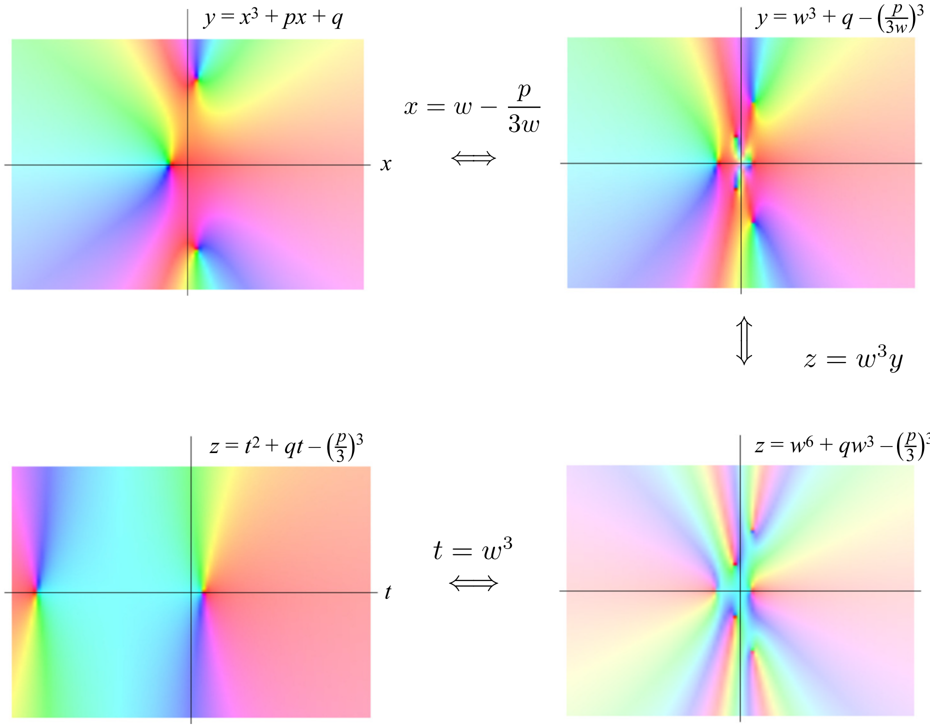

Figure 3. The complex plots of the functions from Figure 1 when $\Delta < 0$.

roots. Two of these roots are on the real axis; they are the two roots of $z = w^6 + qw^3 - (p/3)^3$ shown in Figure 1. The four complex roots in the upper right plot correspond to the complex roots of the cubic. The transformation φ is a two-to-one mapping on the complex plane as well as on the real line. Accordingly, φ takes two roots in the w-plane to one root in the x-plane. As we move between the lower right image and the upper right image, the six roots are fixed since we only modify the outputs and not the w-plane. Since the outputs are modified, the image is colored differently. The transformation pictured in the two lower plots takes the six roots into the two roots of the quadratic.

Positive discriminant. Figure 4 shows the transformation φ in this situation. The transformed curve $y = w^3 + q - (p/(3w))^3$ never crosses the w-axis. Thus, under the transformation, each real root of $y = x^3 + px + q$ is mapped to a complex root of $y = w^3 + q - (p/(3w))^3$, and we cannot provide a real plot visualization analogous to Figure 1—the only way to visualize this situation is the complex plots in Figure 5.

The two upper images in Figure 5 correspond to the graphs in Figure 4. The roots in Figure 5 are labeled 1, 2, and 3. In the upper left image, the three roots lie on the real axis, but the roots are no longer real in the other images.

The lower left image of Figure 5 shows the two (conjugate) complex roots of the quadratic equation $z = t^2 + qt - (p/3)^3$. The image in the lower right shows the six roots of the sixth degree polynomial. Every complex number (except 0) has three complex cube roots, and w is the cube root of t. Thus, the two roots for z (t-values) in the lower left image become the six roots for z (w-values) in the lower right image. Again,

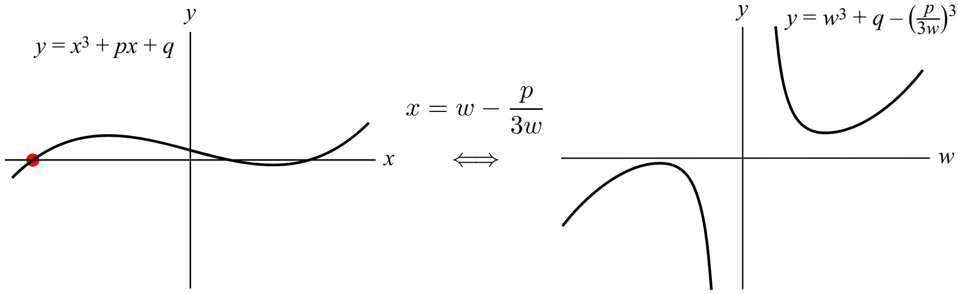

Figure 4. The transformation φ when $\Delta > 0$.

as we move from the lower right image to the upper right, the roots are fixed; the upper right image corresponds to the graph of the right side of Figure 4.

Finally, the reader can verify that for complex $w = c + id$, where $c^2 + d^2 = -p/3$, we have $x = w - p/(3w) = 2c$. Hence, each conjugate pair of roots in the upper right image corresponds to single real root in the upper left image. Thus, we have the three real roots of the cubic. We note that we have made particular choices in following the roots through the transformations because φ is a two-to-one map and θ is a three-to-one map (when considered as a map on the complex plane).

Zero discriminant. This final case has various similarities with the previous two. The real plots are similar to the $\Delta < 0$ situation, so we simply describe the differences. The graph of the depressed cubic is given in Figure 6. The double root is pictured in

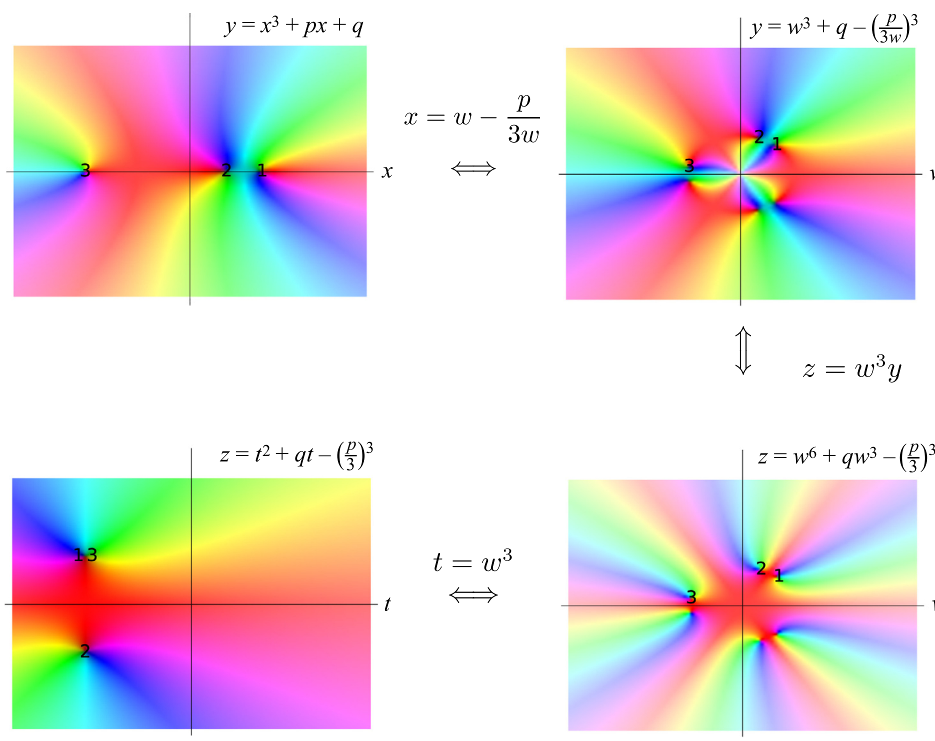

Figure 5. The complex plots of the functions from Figure 1 when $\Delta > 0$.

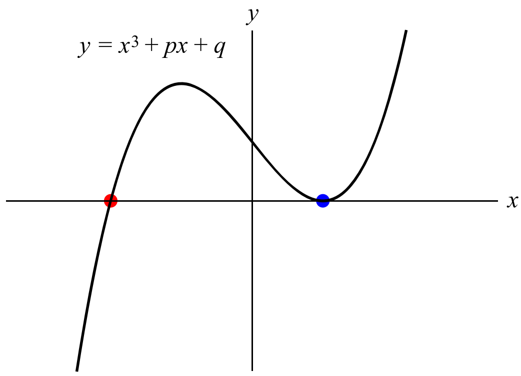

Figure 6. Real plot of depressed cubic with $\Delta = 0$.

blue (darker gray), lying where the graph is tangent to the x-axis. The graph of the corresponding function $z = w^6 + qw^3 - (p/3)^3$ will have the same shape as the one given in the lower right of Figure 1; the only difference is that the minimum point is on the w-axis. In other words, there is only one real w-root, and it is a double root. Similarly, in the graph of the corresponding quadratic $z = t^2 + qt - (p/3)^3$, the minimum point is on the t-axis.

The complex plots in this situation appear similar to those given in in Figure 5, but we include them as Figure 7 for completeness.

The plot in the upper left of Figure 7 contains two real roots instead of three. The images in the upper and lower right (the w-complex plane) now show one real root

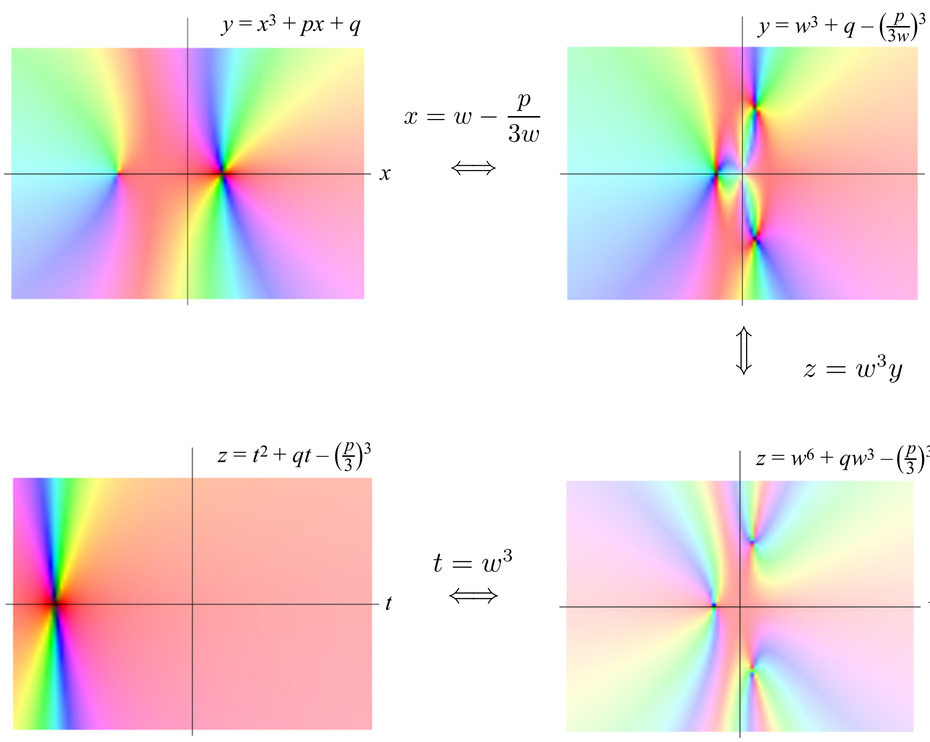

Figure 7. The complex plots of the functions from Figure 1 when $\Delta = 0$.

and two complex roots. Each of these roots represents a double root. This situation can be thought of as a limiting process using the $\Delta > 0$ case: As two real roots move together, the two roots in the upper complex plane move together, the two roots in the lower complex plane move together, and the two roots with negative real part move together. Similar remarks apply to the plot in the lower left of Figure 7, which pictures a double root in the t-plane.

A slightly stronger result

In Figure 5 we see that roots 1 and 3 from the cubic (in the upper left) eventually map to the same root of the quadratic (in the lower left). It turns out that the conjugate of root 2 from the lower right graph also maps to this same root of z. Thus, we actually have the following result, which is a bit stronger than Theorem 1.

Theorem 2. *Let p and q be integers. If a is any solution of the depressed cubic equation $x^3 + px + q = 0$, then a is of the form*

$$\sqrt[3]{\left(-\frac{q}{2}\right) + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\left(-\frac{q}{2}\right) - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

The point here is that *every* root of the cubic can be obtained in this way because the cube root symbol is not unique when the argument is a complex number. When we move from the lower left to the lower right of the complex plots in Figures 3, 5, and 7, we make a choice of cube roots. The different choices allow us to obtain any one of the three roots of the depressed cubic.

Summary. We visually investigate Viète's method for funding for the roots of the depressed cubic equation using transformations and complex plots from Sage to consider the three cases.

References

1. E. A. González-Velasco, *Journey through Mathematics: Creative Episodes in Its History*. Springer, New York, 2011.
2. M. Lunsford, Cardano, *casus irreducibilis*, and finite fields, *Math. Mag.* **87** (2014) 377–380, http://dx.doi.org/10.4169/math.mag.87.5.377.
3. R. W. D. Nickalls, Viète, Descartes and the cubic equation, *Math. Gaz.* **90** (2006) 203–208.
4. The Sage Developers, *Sage Mathematics Software*. Version 7.4. 2017, http://www.sagemath.org.
5. ———, *Sage's Version 7.5 Reference Manual*. 2017, http://www.sagemath.org/doc/reference.
6. I. J. Zucker, The cubic equation—A new look at the irreducible case, *Math. Gaz.* **92** (2008) 264–268.

