ATM 2016 Conference

Using Mapping Diagrams to Make Sense of Functions and Equations

March 31, 2016

EF3 E 9:00-10:30 F 11:00-12:30

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Abstract

Participants will learn how to use mapping diagrams (MD) to make sense of functions and equations.

A mapping diagram is an alternative to a Cartesian graph that visualizes a function. Like a table, it can present finite data, but also can be used dynamically with technology.

An overview of basic function concepts with MD's will begin the session using GeoGebra, followed by connections of MD's to solving linear and quadratic equations.

Abstract

Background and examples are available at Mapping Diagrams from A(lgebra) B(asics) to C(alculus) and D(ifferential) E(quation)s. A Reference and Resource for Function Visualizations Using Mapping Diagrams.
http://users.humboldt.edu/flashman/MD/sec

ATM 2016 Conference

Using Mapping Diagrams to Make Sense of Functions and Equations

Link
to Presentation Files:

http://users.humboldt.edu/flashman/Presentations/ATM/ATM.LINKS.html

to GeoGebra File:

https://www.geogebra.org/apps/?id=JkqNduU9

Background Questions

Thumbs Up or Down…

1. Are you familiar with Mapping Diagrams?
2. Have you used Mapping Diagrams to teach functions?
3. Have you used Mapping Diagrams to teach content besides function definitions?

Mapping Diagrams

A.k.a.

Function Diagrams

Dynagraphs

Preface: Quadratic Example Will be reviewed later.☺

$g(x) = 2\,(x\text{-}1)^2 + 3$

Steps for *g*:

1. **Linear:**

   Subtract 1.

2. **Square result.**

3. **Linear:**

   Multiply by 2

   then add 3.

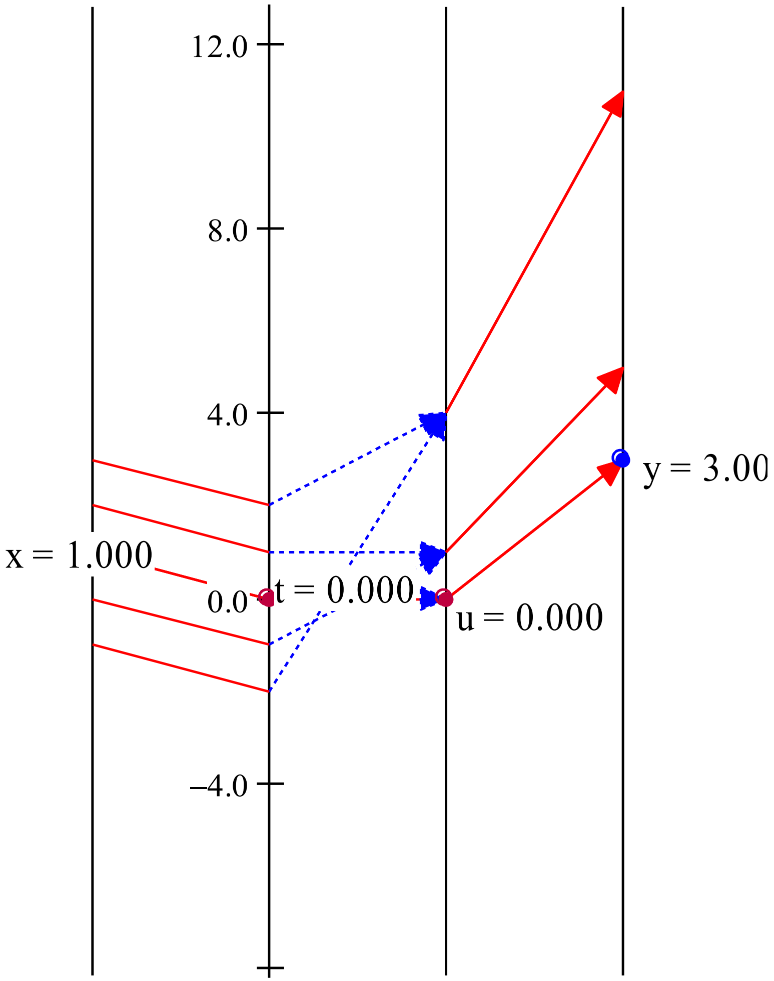
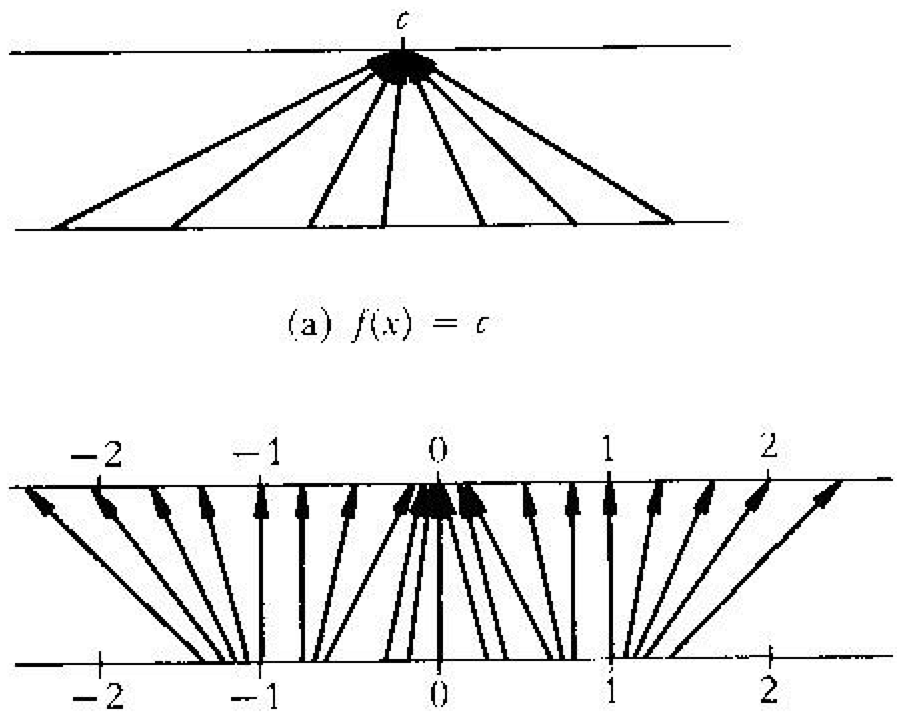


Figure from Ch. 5
Calculus by M. Spivak



(a) $f(x) = c$

(b) $f(x) = x^3$

FIGURE 2

Main Resource

- Mapping Diagrams from A(lgebra) B(asics) to C(alculus) and D(ifferential) E(quation)s. A Reference and Resource Book on Function Visualizations Using Mapping Diagrams (Preliminary Sections- NOT YET FOR publication)

- http://users.humboldt.edu/flashman/MD/section-1.1VF.html

Old Friends:
Linear Function Examples

i. $m(x) = mx; \quad m = 2$

ii. $s(x) = x + b; \quad b = 1$

iii. $f(x) = mx + b$

$= s(m(x))$

$= 2x + 1$

Distribute Worksheet

Thumbs up when you are ready to proceed.

Old Friends: Linear Function Examples

- Worksheet 1.a
- Make tables for $m(x) = 2x$ and $s(x) = x+1$

x	m(x) =2x
2	
1	
0	
-1	
-2	

x	s(x) =x+1
2	
1	
0	
-1	
-2	

Function Tables

- Worksheet 1.a
- Make tables for $m(x) = 2x$ and $s(x) = x + 1$

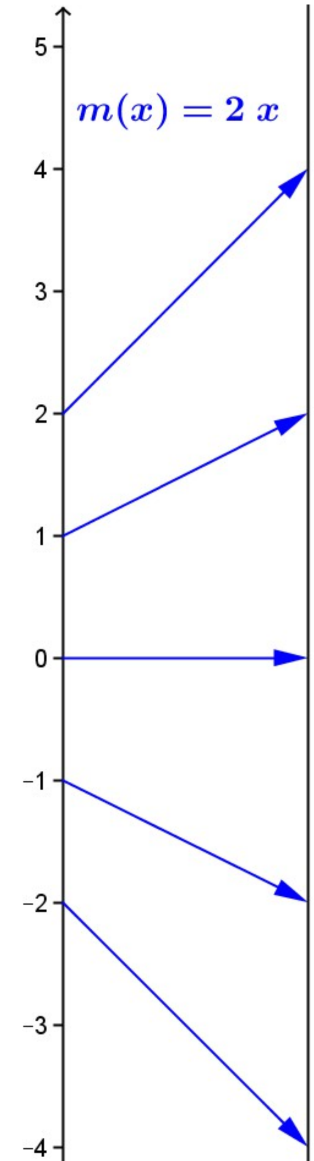
x	m(x) =2x
2	4
1	2
0	0
-1	-2
-2	-4

x	s(x) =x+1
2	3
1	2
0	1
-1	0
-2	-1

Worksheet 1.b

Mapping Diagram for m(x) = 2x

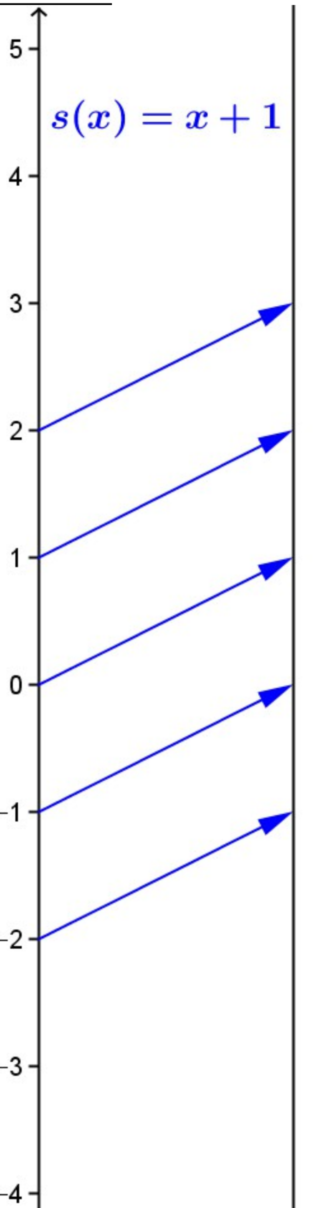
x	m(x) =2x
2	4
1	2
0	0
-1	-2
-2	-4



Worksheet 1.b

Mapping Diagram for s(x) = x +1

x	s(x)=x+1
2	3
1	2
0	1
-1	0
-2	-1



Mapping Diagram Prelim

- Examples of mapping diagrams
  - Worksheet 2
  - a. First make table for $q(x) = x^2$.

x	$q(x) = x^2$
2	
1	
0	
-1	
-2	

Mapping Diagram Prelim

- Examples of mapping diagrams
  - Worksheet 2
  - a. First make table for q.

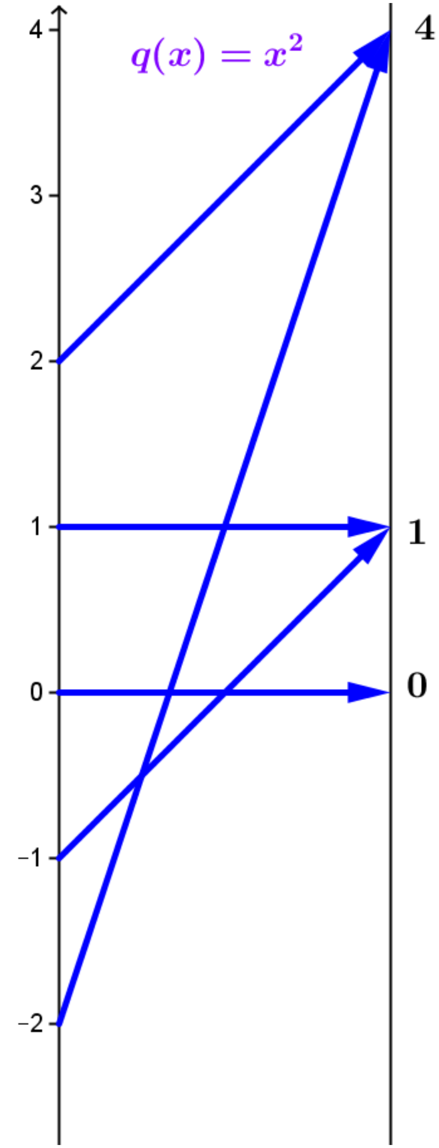
x	$q(x) = x^2$
2	4
1	1
0	0
-1	1
-2	4

  - b. Sketch a mapping diagram for $q(x) = x^2$.

Mapping Diagram Prelim

Worksheet 2.b. Mapping Diagram for $q(x) = x^2$

x	$q(x) = x^2$
2	4
1	1
0	0
-1	1
-2	4



Worksheet 3.a.Complete the following table for the composite function $f(x) = s(m(x)) = 2x + 1$

x	m(x)	f(x)=s(m(x))
2		
1		
0		
-1		
-2		

Worksheet 3.a.Complete the following table for the composite function f(x) = s(m(x)) = 2x +1

x	m(x)	f(x)=s(m(x))
2	4	5
1	2	3
0	0	1
-1	-2	-1
-2	-4	-3

Mapping Diagram Prelim

- Worksheet 3.b
- Use the table 3.a and the previous sketches of 1.b to draw a composite sketch of the mapping diagram with 3 axes for the composite function

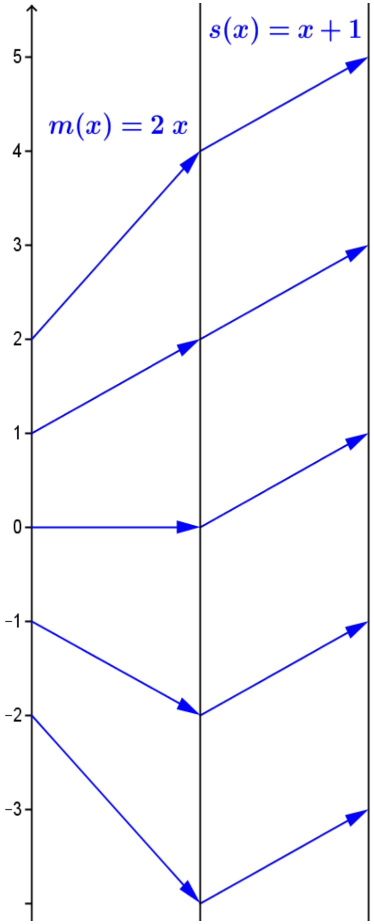
$$f(x) = h(g(x)) = 2x + 1$$

Worksheet 3.b Draw a sketch for the mapping diagram with 3 axes of $f(x) = 2x + 1$.

x	m(x)	f(x)=s(m(x))
2	4	5
1	2	3
0	0	1
-1	-2	-1
-2	-4	-3

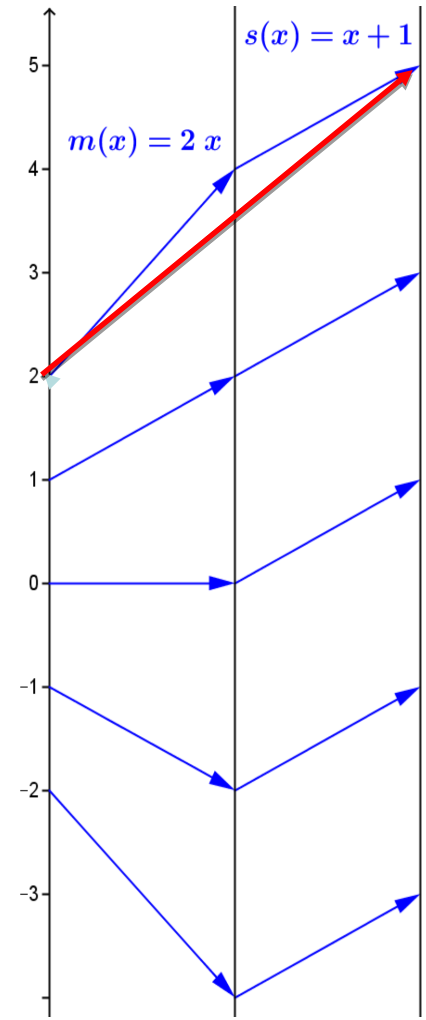
Worksheet 3.b Draw a sketch for the mapping diagram with 3 axes of $f(x) = 2x + 1$.

x	m(x)	f(x)=s(m(x))
2	4	5
1	2	3
0	0	1
-1	-2	-1
-2	-4	-3



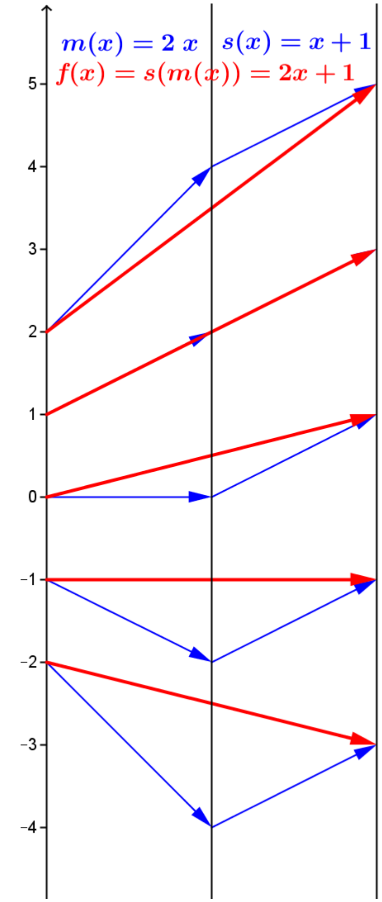
Worksheet 3.c Draw a sketch for the mapping diagram with 2 axes of $f(x) = 2x + 1$.

x	m(x)	f(x)=s(m(x))
2	4	5
1	2	3
0	0	1
-1	-2	-1
-2	-4	-3



Mapping Diagram for f(x) = s(m(x)) = 2x +1

x	m(x)	f(x)=s(m(x))
2	4	5
1	2	3
0	0	1
-1	-2	-1
-2	-4	-3



Technology Examples

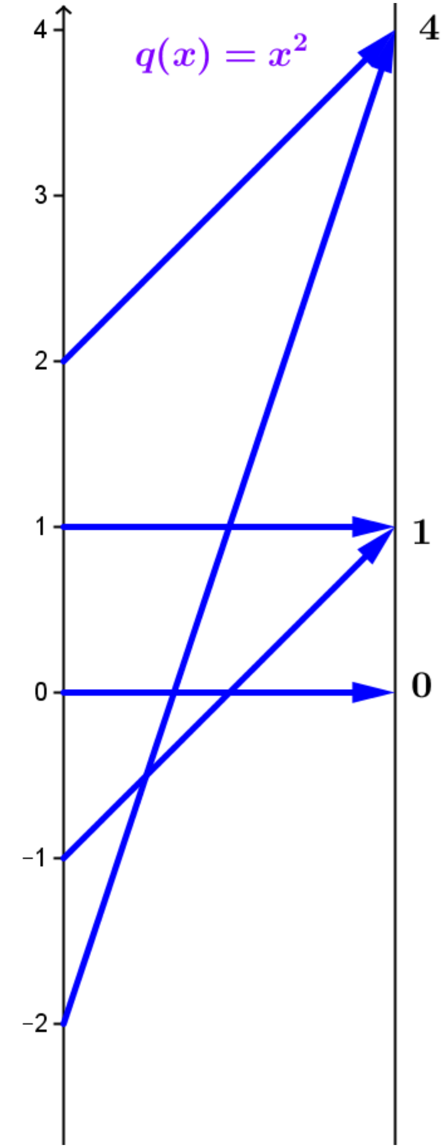
- Excel example
- **Link to GeoGebra File**

  https://www.geogebra.org/apps/?

  –

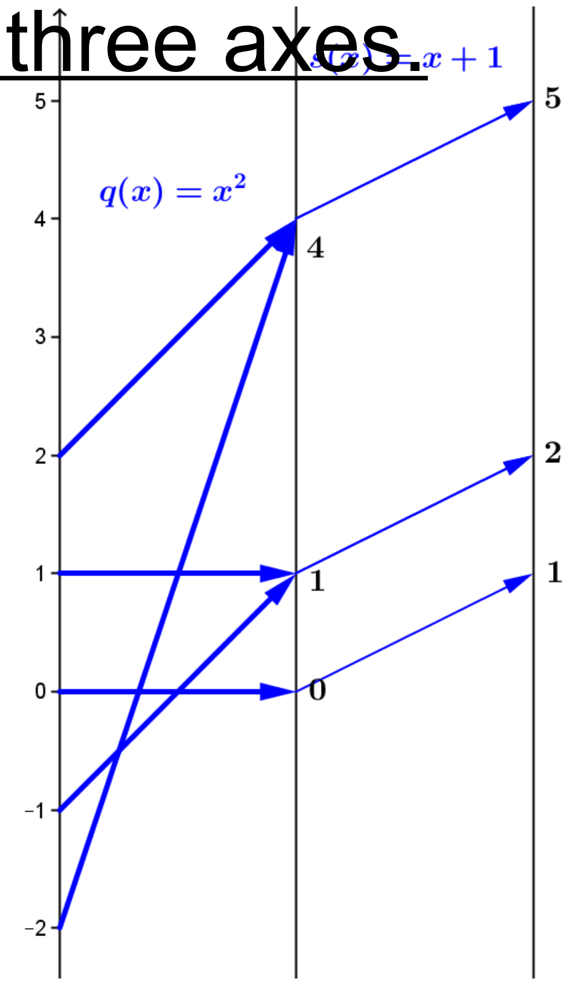
Worksheet 4 Mapping Diagram:
$q(x) = x^2$

x	$q(x) = x^2$
2	4
1	1
0	0
-1	1
-2	4



Worksheet 4.b

- 4.b Using the data from part a), sketch mapping diagrams for the composition $R(x) = s(q(x)) = x^2 + 1$ with three axes.



Worksheet 4.b

- 4.b Using the data from part a), sketch mapping diagrams for the composition $R(x) = s(q(x)) = x^2 + 1$ with two axes.

*** Part I ***
Mapping Diagrams and Solving A Linear Equation.

An Old Friend: Solving A Linear Equation

- Worksheet 5.a Solve a linear equation:

$$2x + 1 = 5$$

Find x.

An Old Friend:
Solving A Linear Equation

Worksheet 5.a Solve a linear equation:

$$2x + 1 = 5$$

$$-1 = -1$$

$$2x = 4$$

An Old Friend: Solving A Linear Equation

Worksheet 5.a Solve a linear equation:

$$2x + 1 = 5$$

$$-1 = -1$$

$$2x = 4$$

$$1/2(2x) = 1/2(4)$$

$$x = 2$$

An Old Friend: Solving A Linear Equation

- Worksheet 5.a Solve a linear equation:

$$2x + 1 = 5$$

$$-1 = -1$$

$$2x = 4$$

$$1/2(2x) = 1/2(4)$$

$$x = 2$$

Check!

$$2x+1 = 2*2 + 1 = 5$$

Linear Equations Use Linear Functions!

Linear Equations

$2x + 1 = 5$

$-1 = -1$

$2x = 4$

$1/2(2x) = 1/2(4)$

$x = 2$

Check:

$2x + 1 = 2*2 + 1 = 5$

Linear Functions

$f(x) = 2x + 1$

Linear Equations Use Linear Functions!

Linear Equations

2x + 1 = 5

-1 = -1

2x = 4

1/2(2x) =1/2(4)

x = 2

Check:

2x + 1 = 2*2 + 1 = 5

Linear Functions

f(x) = 2x + 1

m(x) = 2x; s(x) = x +1

f(x) = s(m(x))

Worksheet 5.b Solving 2x + 1 = 5 visualized with a mapping diagram

Algebra:

$2x + 1 = 5$

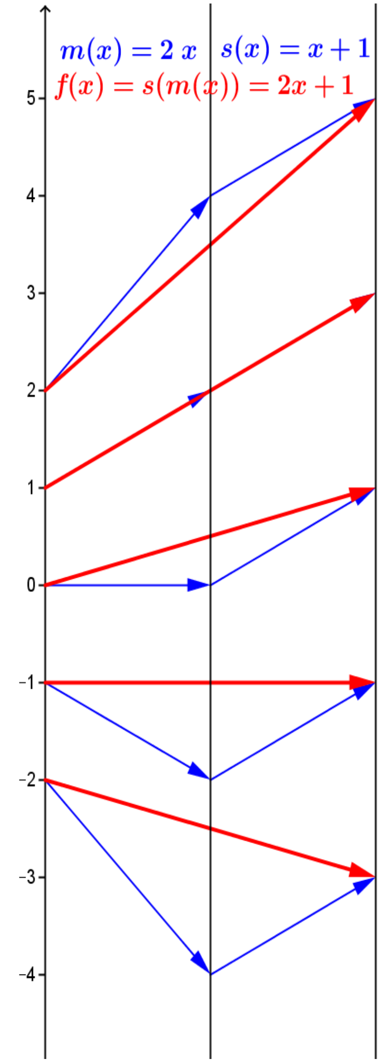
$-1 = -1$

$2x = 4$

$1/2(2x) = 1/2(4)$

$x = 2$

How does the MD for the function VISUALIZE the algebra?



Worksheet 5.b Solving 2x + 1 = 5 visualized with a mapping diagram

Algebra:

$2x + 1 = 5$

$-1 = -1$

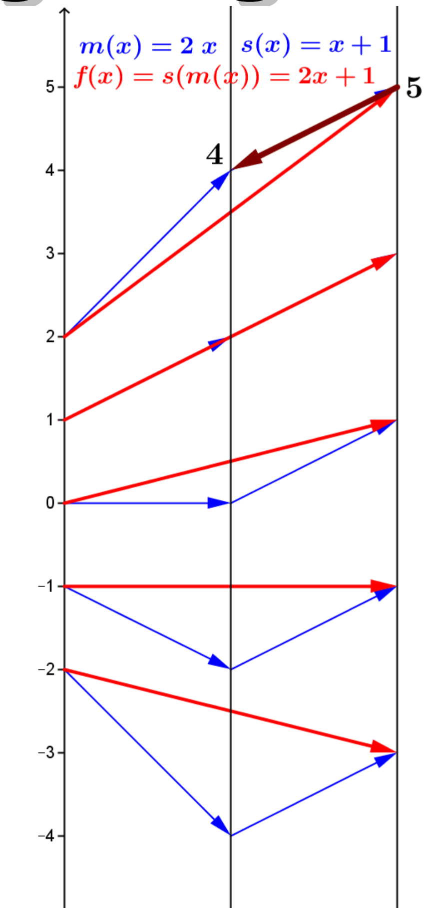
$2x = 4$

Function:

$f(x) = s(m(x)) = 5$

"Undo s"

$m(x) = 4$



Worksheet 5.b Solving 2x + 1 = 5 visualized with a mapping diagram

Algebra:

2x + 1 = 5

-1 = -1

2x = 4

1/2(2x) =1/2(4)

x = 2

Function:

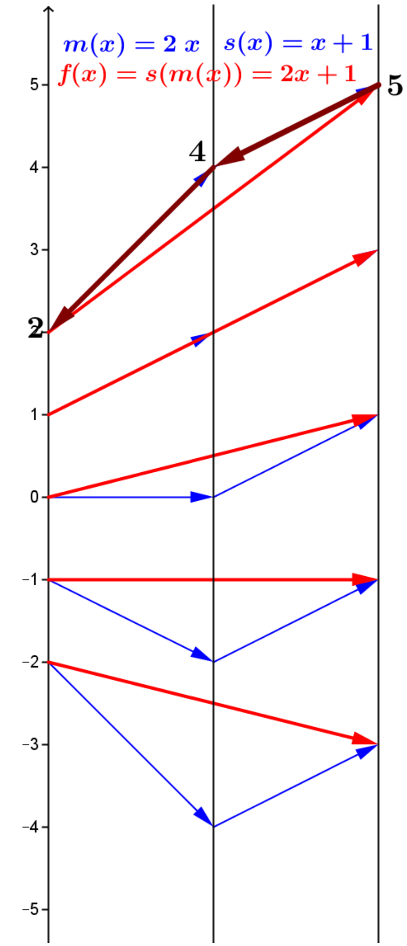
f(x)=s(m(x)) = 5

"Undo s"

m(x) = 4

"Undo m"

x = 2



Worksheet 5.b Solving 2x + 1 = 5 visualized with a mapping diagram

Algebra:

2x + 1 = 5

-1 = -1

2x = 4

1/2(2x) =1/2(4)

x = 2

Function:

f(x)=s(m(x)) = 5

"Undo s"

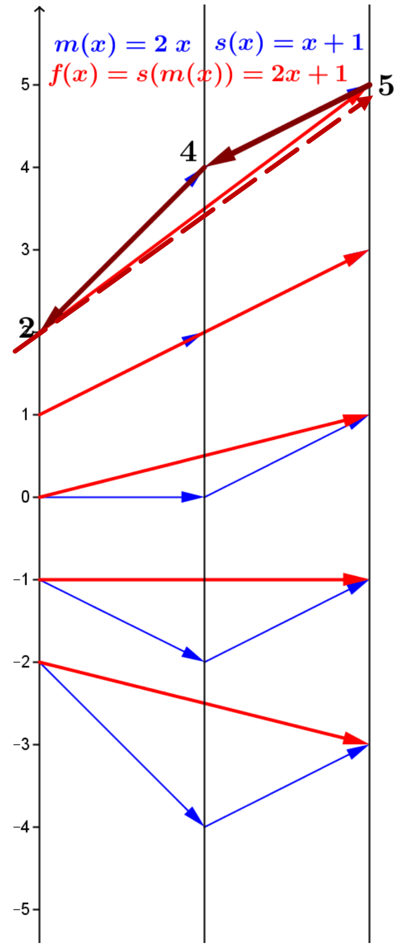
m(x) = 4

"Undo m"

x = 2

CHECK! ☺

$f(2) \overset{!}{=} 5$



Worksheet 5.b Solving 2x + 1 = 5 visualized on GeoGebra

Algebra:

$2x + 1 = 5$

$-1 = -1$

$2x = 4$

$1/2(2x) = 1/2(4)$

$x = 2$

Function:

$f(x)=s(m(x)) = 5$

"Undo s"

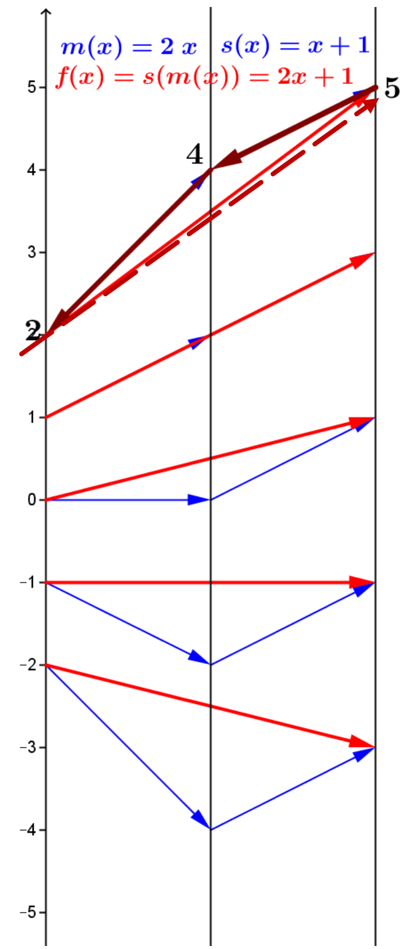
$m(x) = 4$

"Undo m"

$x = 2$

CHECK! ☺

$f(2) \overset{!}{=} 5$



End of Part I
Mapping Diagrams and Solving A Linear Equation.

*** Part II ***
Mapping Diagrams and Solving A Quadratic Equation.

Worksheet 6.a Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Understand the problem

Pause for Discussion.

Worksheet 6.a Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Understand the problem

- $2(x-3)^2 + 1$ is a function of x.
  - $P(x) = 2(x-3)^2 + 1$
- Find any and all x where P(x) = 9.
- $2(x-3)^2 + 1$ is a composition of functions
  - P(x) = s(m(q(z(x)))) where
  - z(x) =
  - q(x) =
  - m(x) =
  - s(x) =

Worksheet 6.a Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Understand the problem

- $2(x-3)^2 + 1$ is a function of x.
  - $P(x) = 2(x-3)^2 + 1$
- Find any and all x where $P(x) = 9$.
- $2(x-3)^2 + 1$ is a composition of functions
  - $P(x) = s(m(q(z(x))))$ where
  - $z(x) = x-3$;
  - $q(x) = x^2$;
  - $m(x) = 2x$;
  - $s(x) = x+1$.

Worksheet 6.a Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram.

Make a plan

– Find any and all x where $P(x) = 9$.

– Construct mapping diagram for P as a composition of function :

$$P(x) = s(m(q(z(x))))$$

– Undo $P(x) = 9$ by undoing each step of P

- Undo $s(x) = x+1$
- Undo $m(x) = 2x$
- Undo $q(x) = x^2$
- Undo $z(x) = x-3$

– Check results to see that $P(x) = 9$

Worksheet 6.b Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram. Execute the **plan**

– Construct mapping diagram for P as a composition of function :

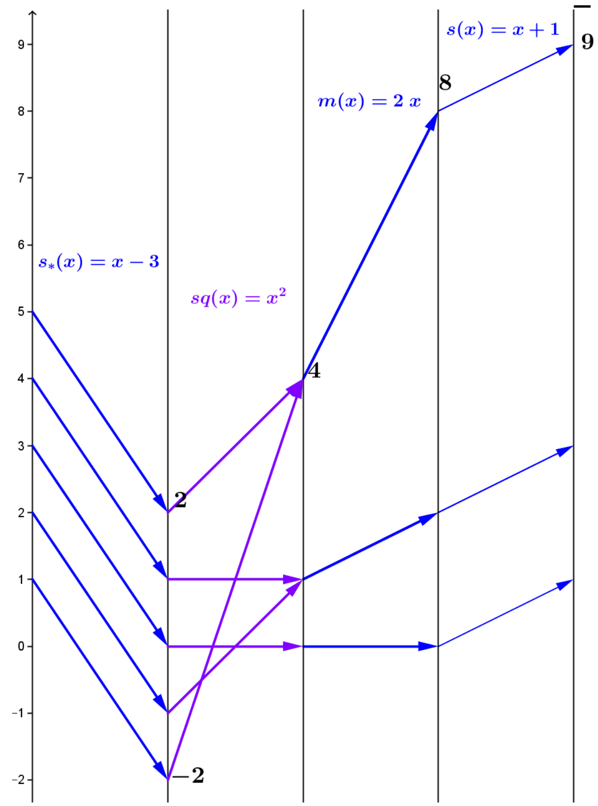
$$P(x) = s(m(q(z(x))))$$

Worksheet 6.b Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram.

Execute the **plan**

- Construct mapping diagram for P as a composition of function :

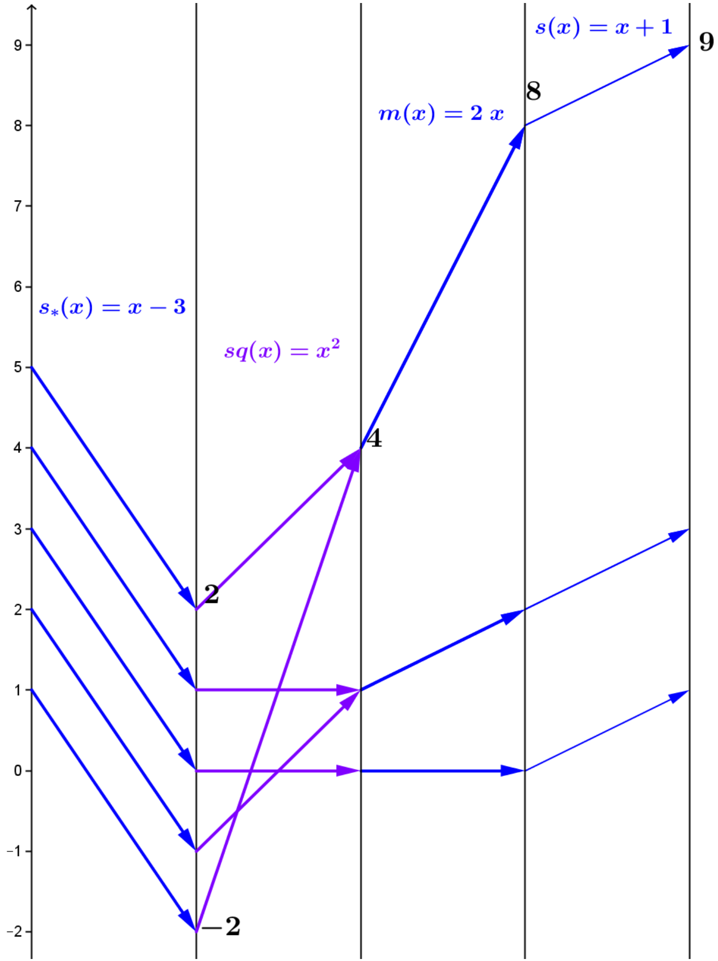
$$P(x) = s(m(q(s_*(x))))$$



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Execute the plan

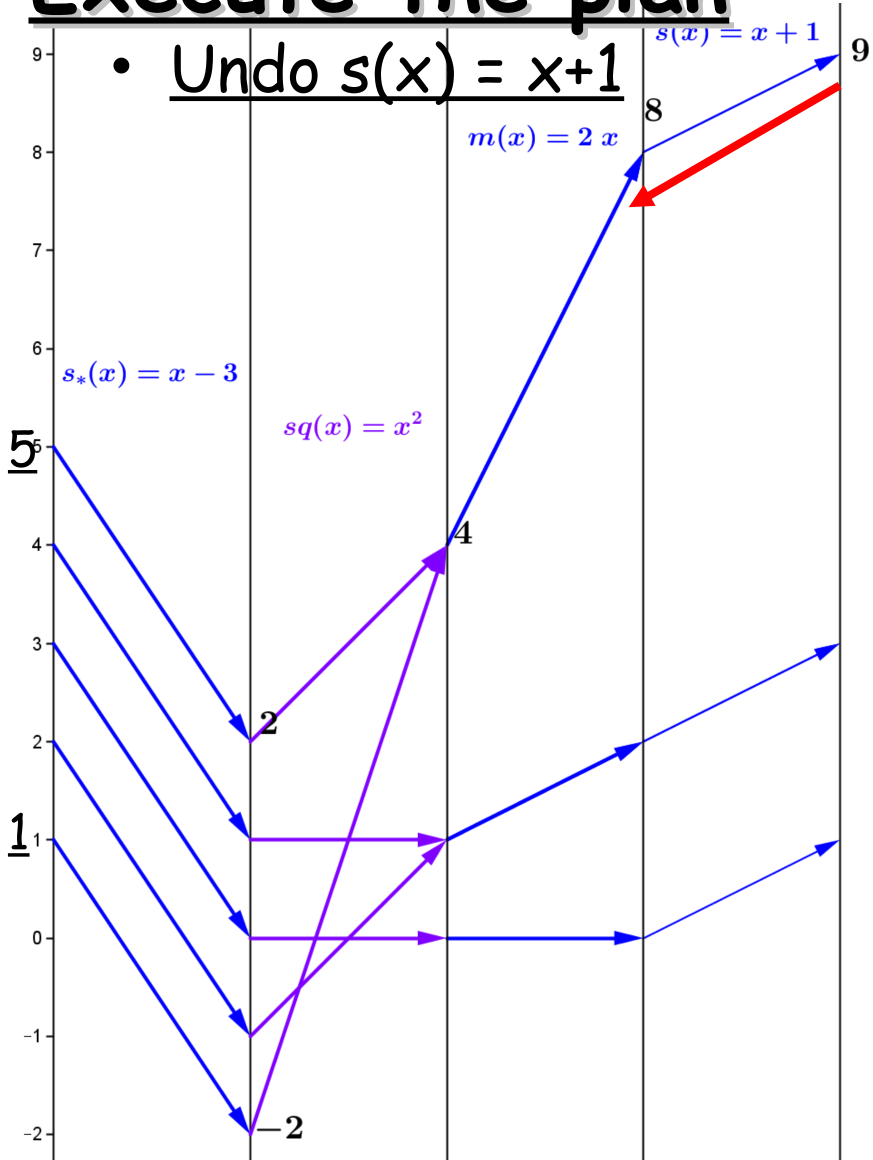
- Find any and all x where P(x) = 9.



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Execute the plan

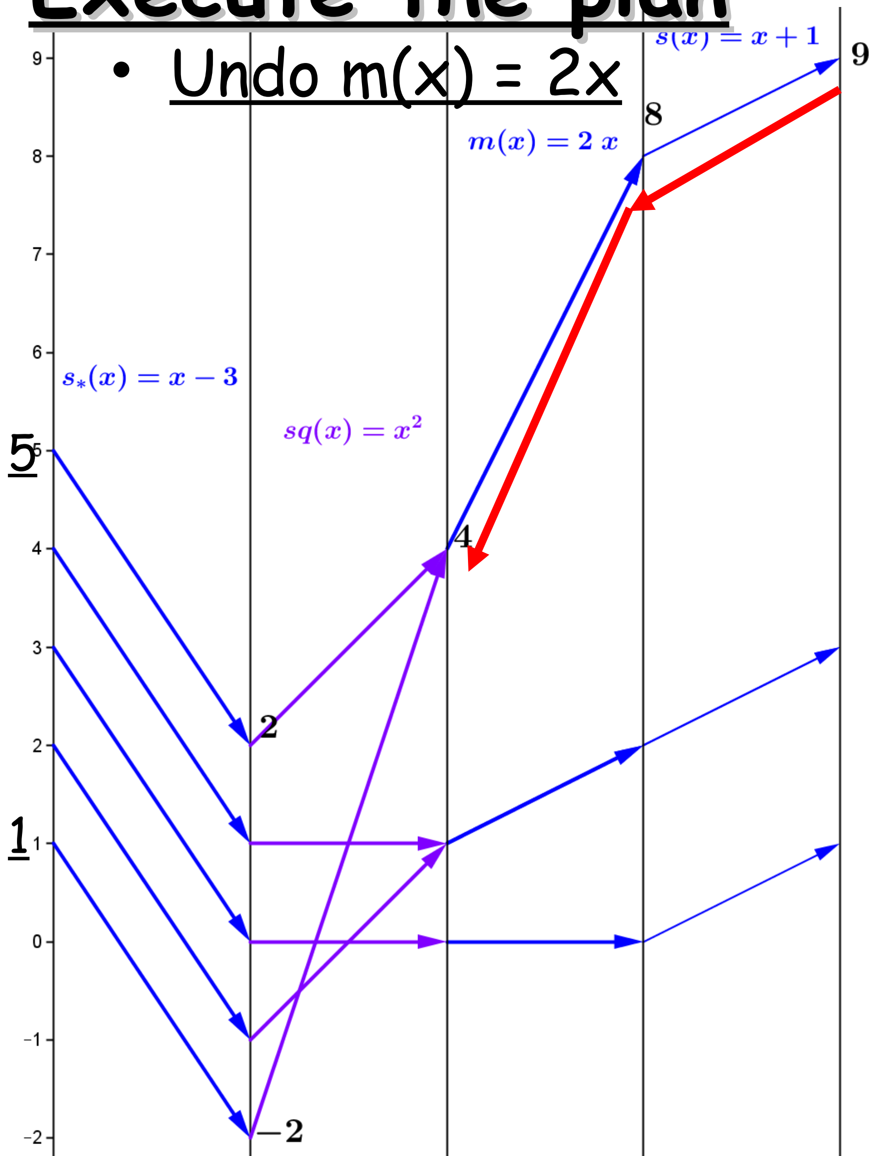
- Undo $s(x) = x+1$



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Execute the plan

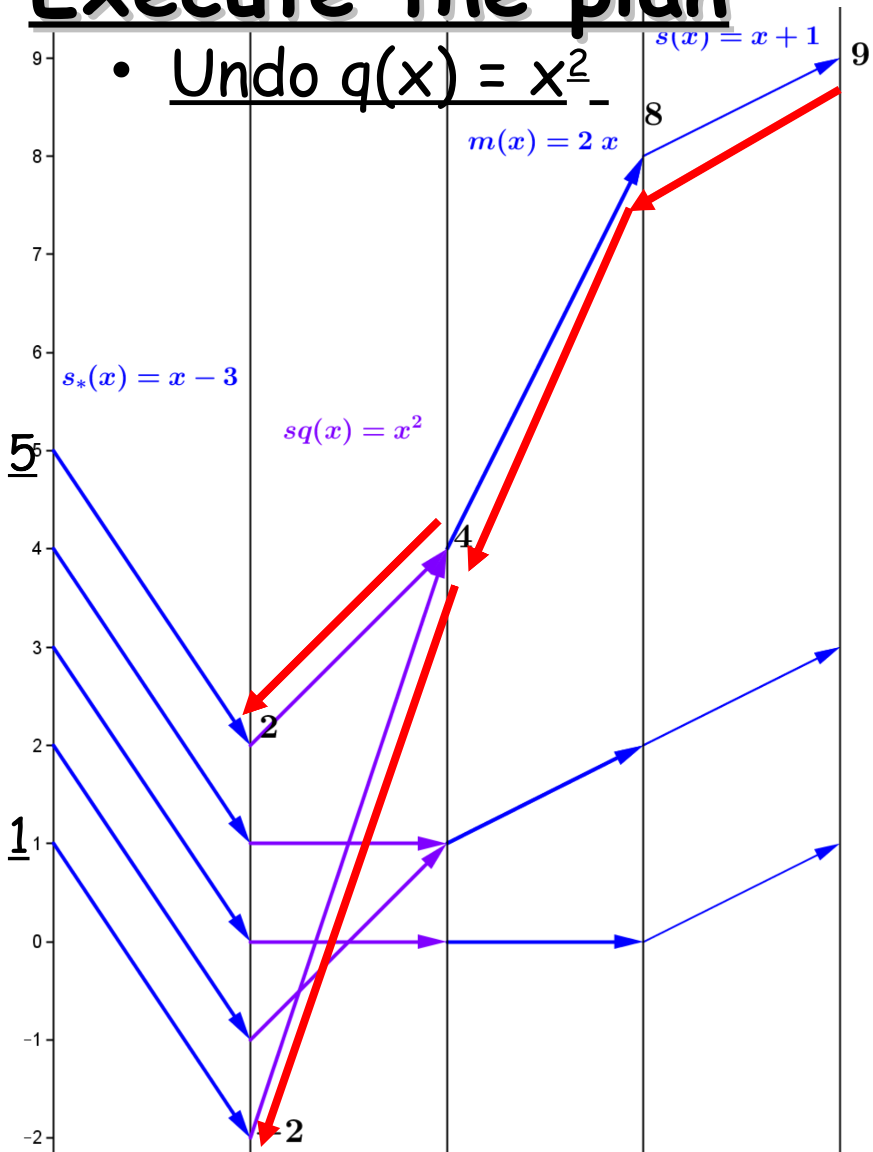
- Undo $m(x) = 2x$



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Execute the plan

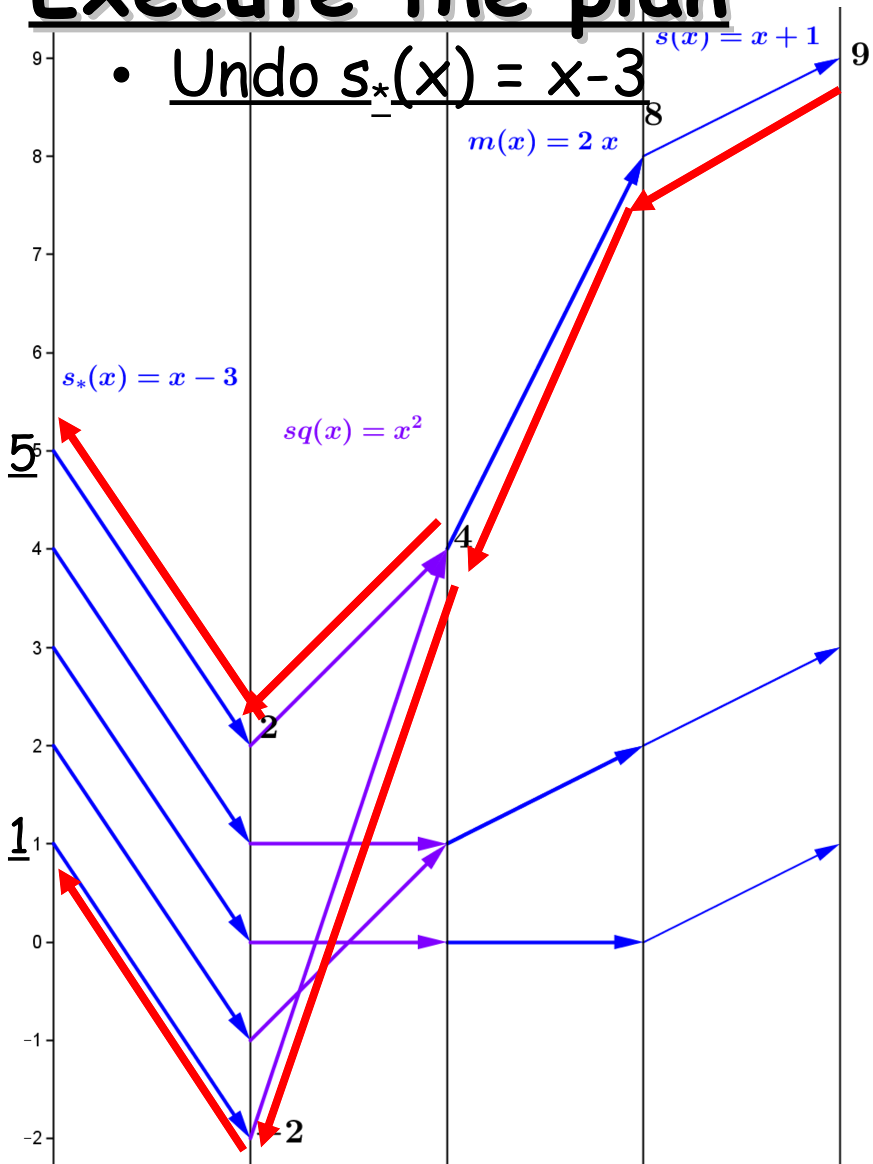
- Undo $q(x) = x^2$



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

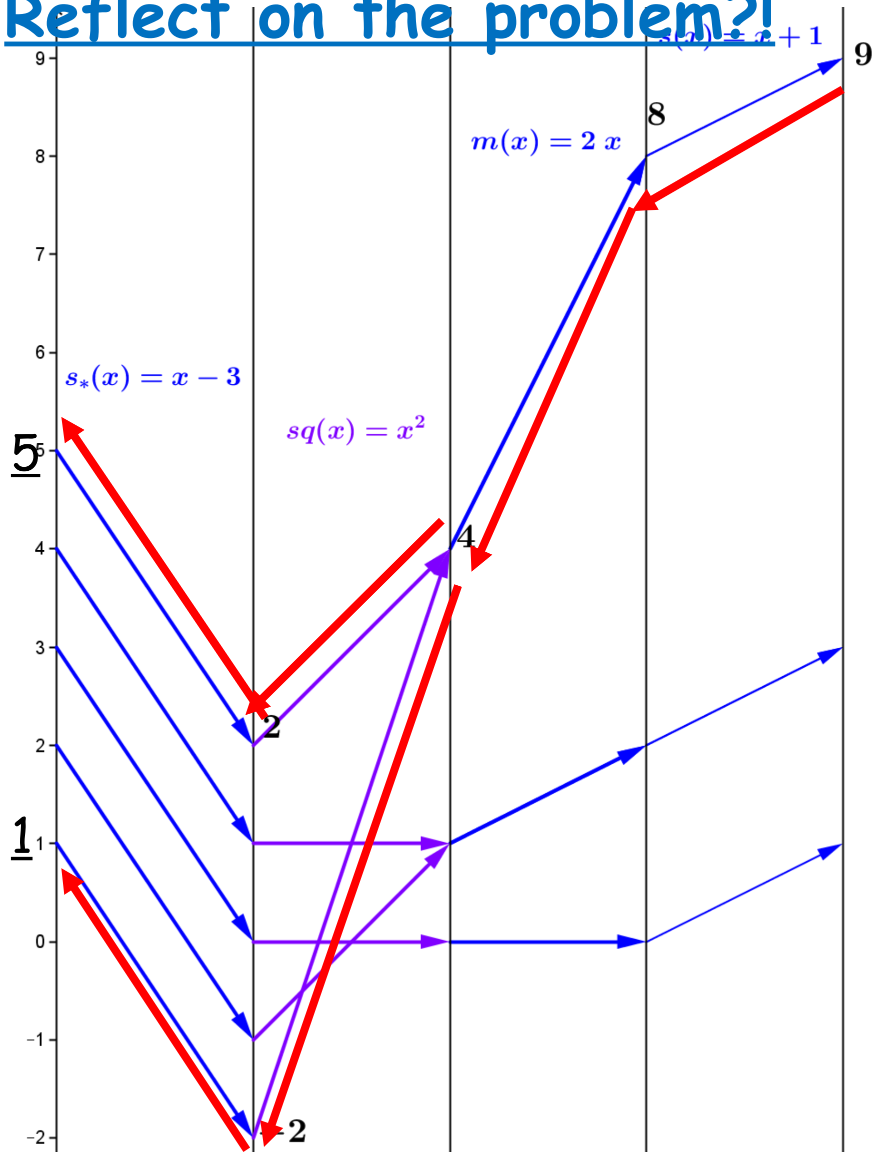
Execute the plan

- Undo $s_*(x) = x-3$



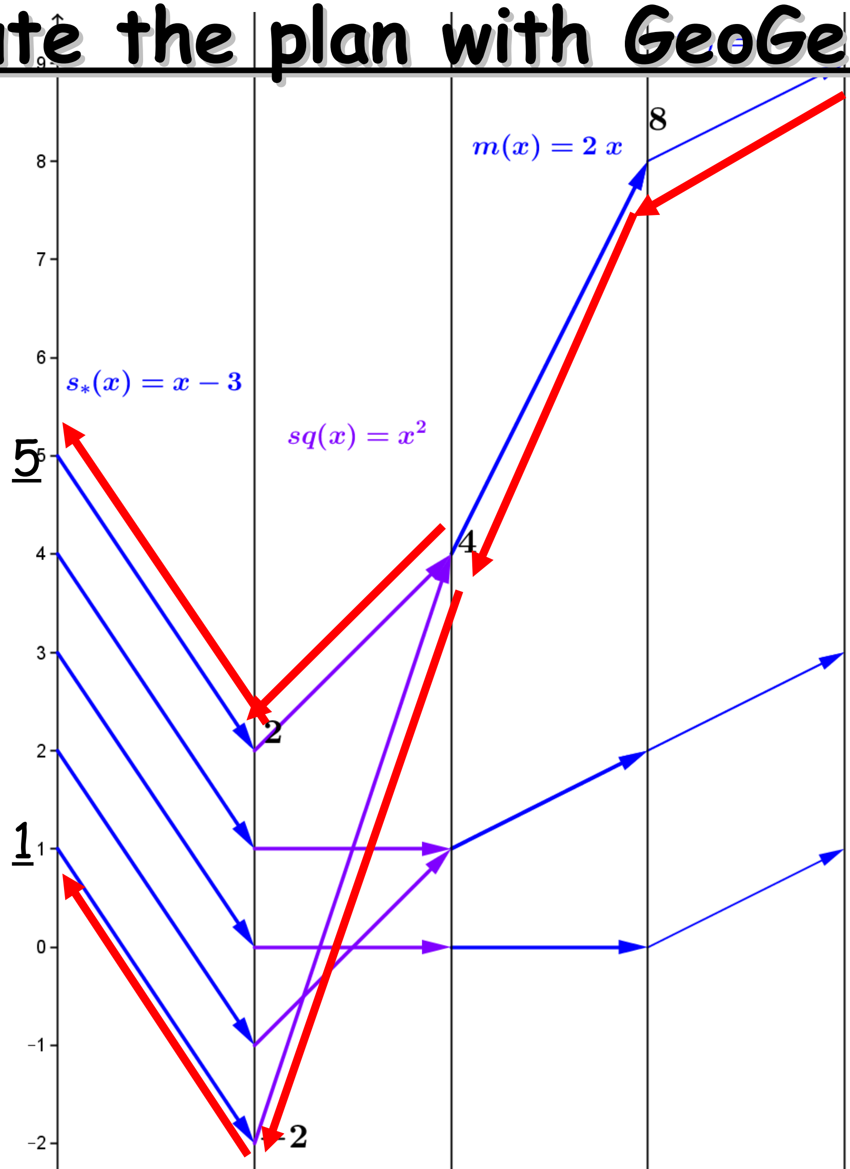
Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Reflect on the problem?!



Worksheet 6.c Solve $2(x-3)^2 + 1 = 9$ with a mapping diagram

Execute the plan with GeoGebra



*** Part III ***
A Second Approach: Linear Mapping Diagrams and Solving Linear Equations.

Simple Examples are important!

- f(x) = x + C Added value: C
- f(x) = mx   Scalar Multiple: m
  Interpretations of m:
  - slope
  - rate
  - Magnification factor
  - m > 0 : Increasing function
  - m < 0 : Decreasing function
  - m = 0 : Constant function

Simple Examples are important!

f(x) = mx + b with a mapping diagram --
Five examples:
Back to Worksheet Problem #7

- Example 1: m =-2; b = 1: f(x) = -2x + 1
- Example 2: m = 2; b = 1: f(x) = 2x + 1
- Example 3: m = ½; b = 1: f(x) = ½ x + 1
- Example 4: m = 0; b = 1: f(x) = 0 x + 1
- Example 5: m = 1; b = 1: f(x) = x + 1

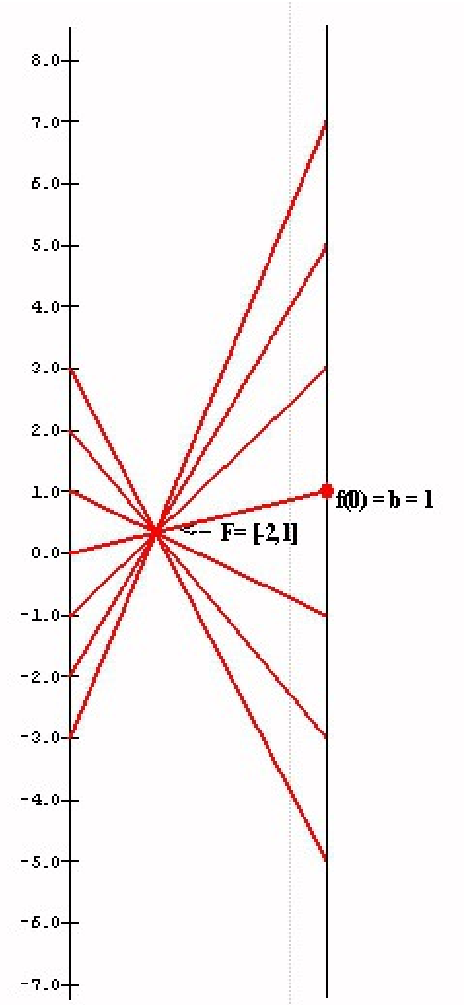
Visualizing $f(x) = \mathbf{m}x + \mathbf{b}$ with a mapping diagram -- Five examples:

Example 1: m = -2; b = 1

$f(x) = -2x + 1$

- Each arrow passes through a single point, which is labeled **F = [- 2,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $-2x + 1$,

    which corresponds to the linear function's value for the point/number, *x*.



Visualizing *f* (*x*) = **m***x* + **b** with a mapping diagram -- Five examples:

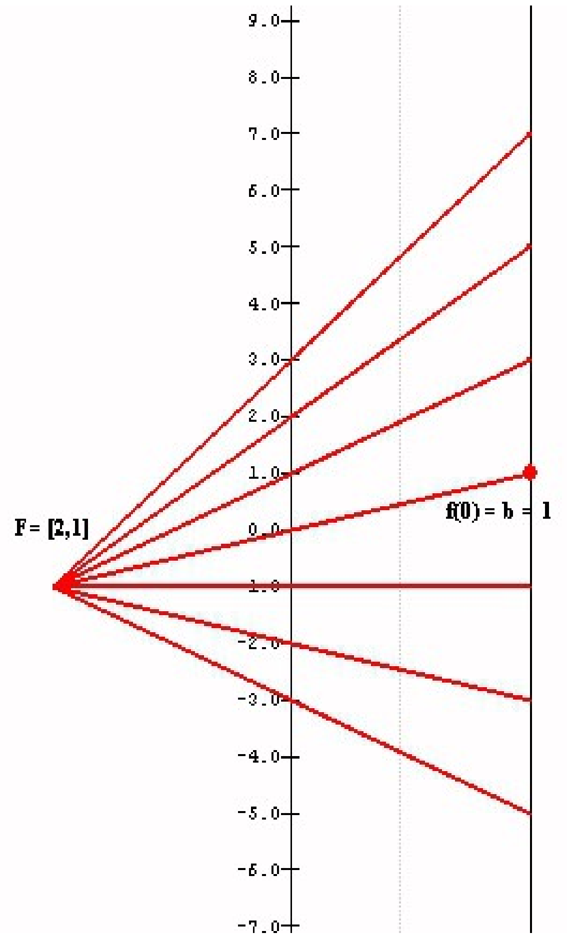
Example 2: *m* = 2; *b* = 1
f(x) = 2x + 1

Each arrow passes through a single point, which is labeled

F = [2,1].

- The point **F** completely determines the function *f*.
  - **given a point** / number, *x*, on the source line,
  - there is a **unique arrow passing through F**
  - **meeting** the target line at **a unique point** / number, 2*x* + 1,

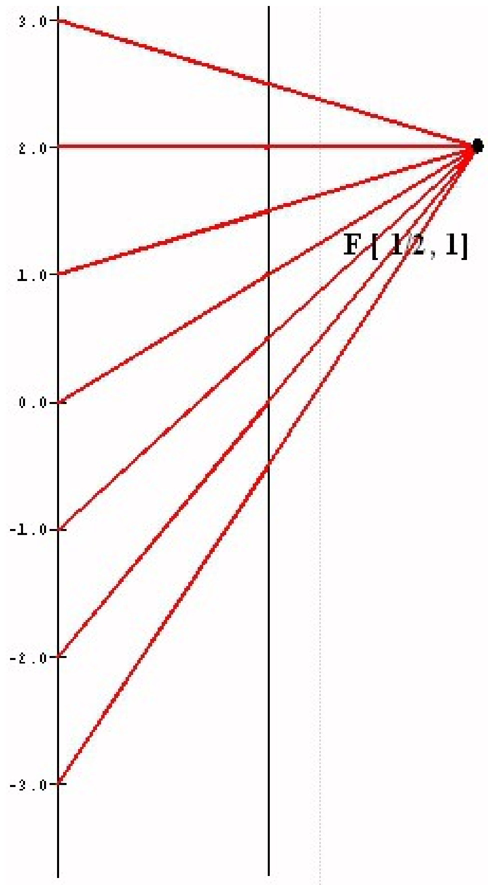
  which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = mx + b$ with a mapping diagram -- Five examples:

- **Example 3: $m = 1/2$; $b = 1$**
  $f(x) = \frac{1}{2}x + 1$
- Each arrow passes through a single point, which is labeled **F = [1/2,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $\frac{1}{2}x + 1$,

    which corresponds to the linear function's value for the point/number, *x*.

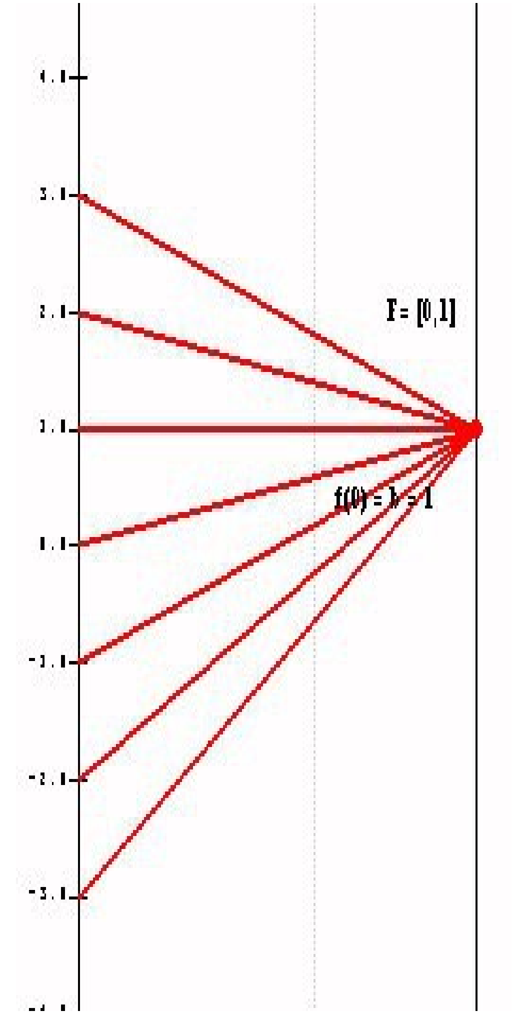


Visualizing $f(x) = \boldsymbol{m}x + \boldsymbol{b}$ with a mapping diagram -- Five examples:

- **Example 4: m = 0; b = 1**
  f(x) = 0 x + 1

- Each arrow passes through a single point, which is labeled **F = [0,1]**.
  - The point **F** completely determines the function f.
    - **given a point** / number, x, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $f(x)$=1,

    which corresponds to the linear function's value for the point/number, x.



Visualizing *f* (*x*) = ***m**x* + ***b*** with a mapping diagram -- Five examples

Example 5: ***m*** = 1; ***b*** = 1

f (*x*) = *x* + ***1***

- Unlike the previous examples, in this case it is not a single point that determines the mapping diagram, but the single arrow from 0 to 1, which we designate as F[1,1]
- It can also be shown that this single arrow completely determines the function. Thus, given a point / number, *x*, on the source line, there is a unique arrow passing through ***x parallel to*** F[1,1] meeting the target line a unique point / number, *x* + 1, which corresponds to the linear function's value for the point/number, *x*.
    - The single arrow completely determines the function *f*.
        - **given a point** / number, *x*, on the source line,
        - there is a **unique arrow** through ***x parallel to*** F[1,1]
        - **meeting** the target line at **a unique point** / number, *x* + 1,

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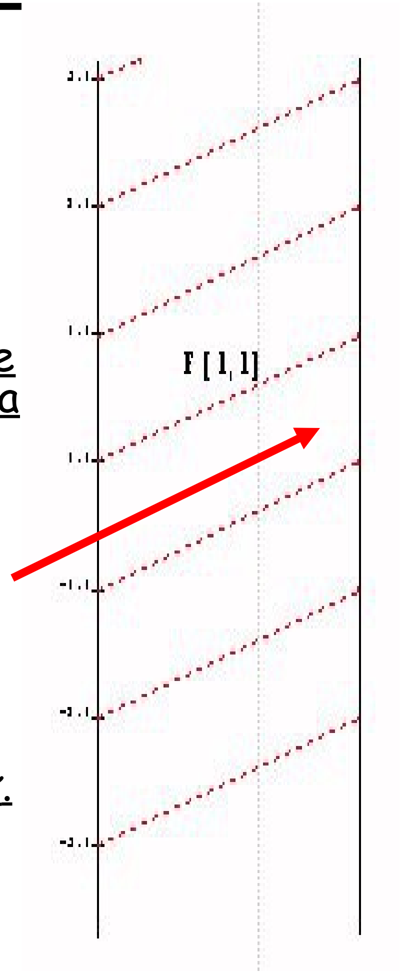

Function-Equation Questions
with linear focus points (Problem 8.a)

- Use a focus point in the mapping diagram to solve a linear equation:

$$2x+1 = 5$$

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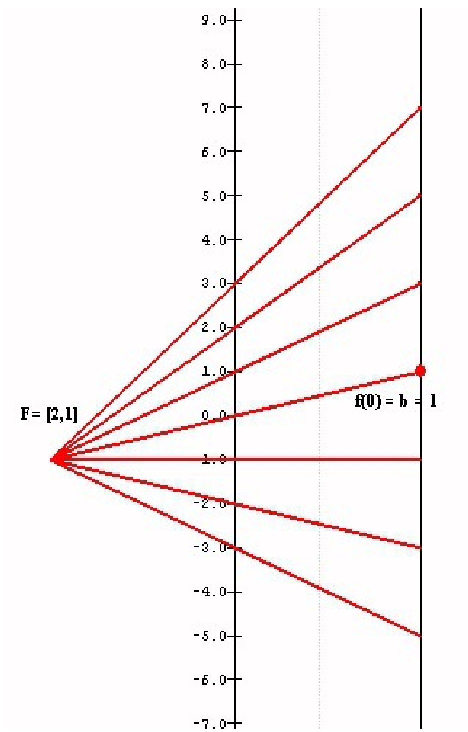

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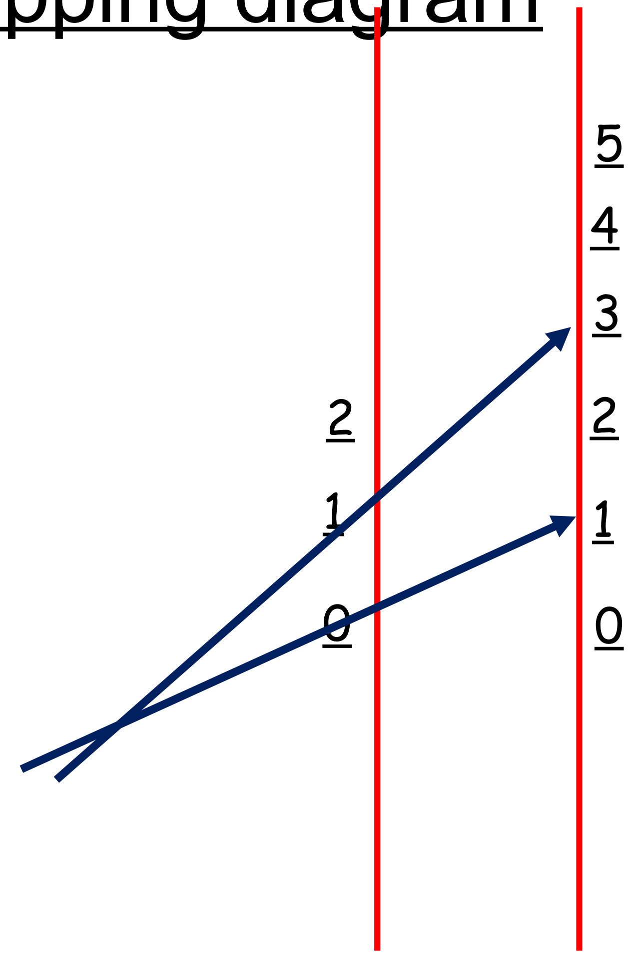

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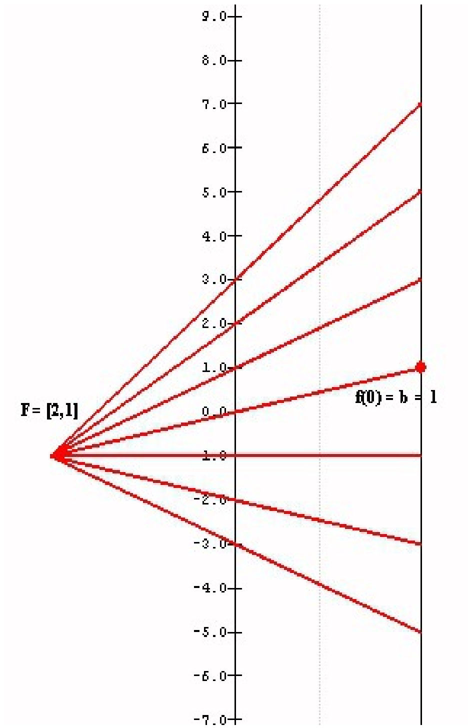

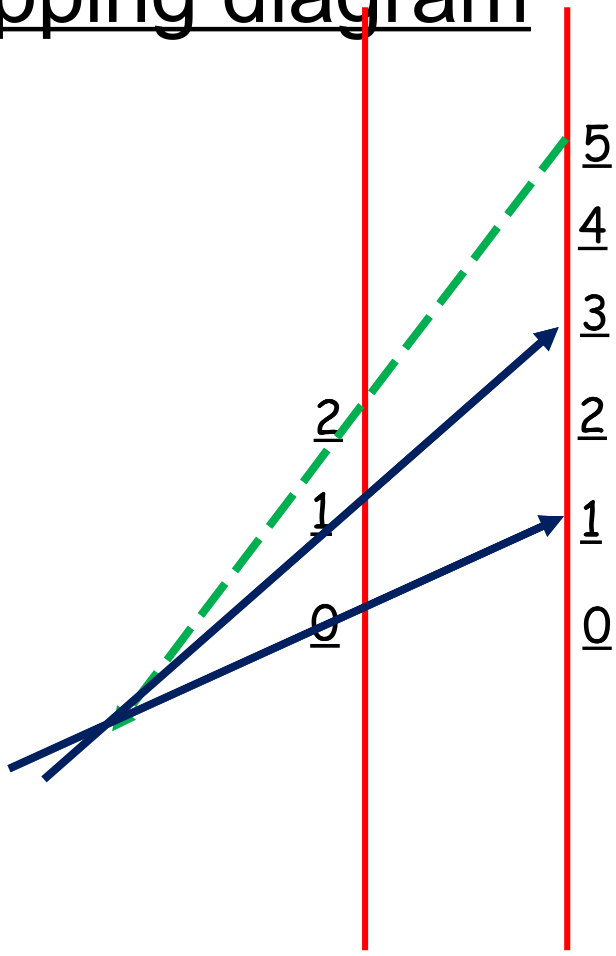

Function-Equation Questions

with linear focus points (Problem 8)

Suppose f is a linear function with f (1) = 3 and f (3) = -1.

Without algebra

- 8.b Use a focus point to find f (0).
- 8.c Use a focus point to find x where f (x) = 0.

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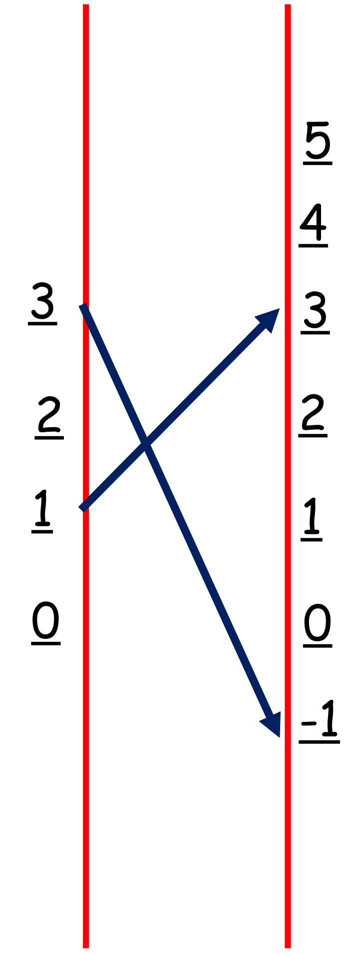

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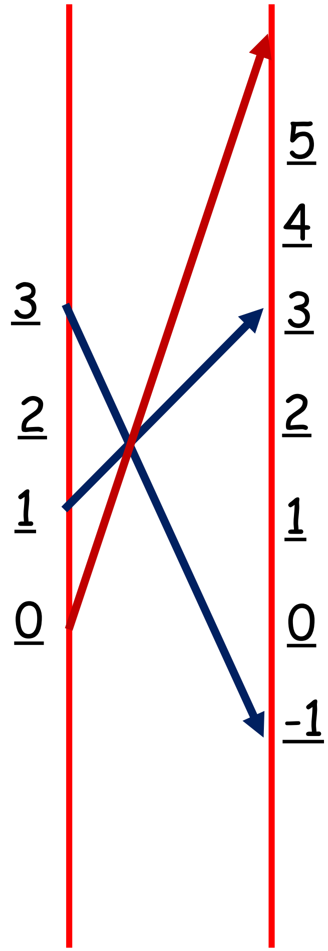

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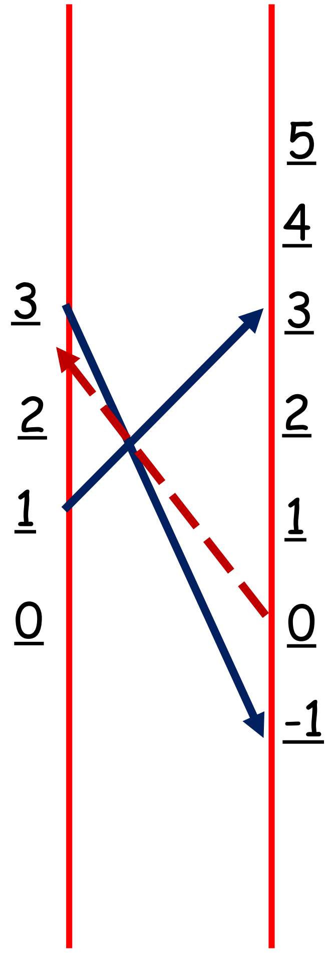


*** End of Part III ***

A Second Approach: Linear Mapping Diagrams and Solving Linear Equations.

*** Part IV ***
A Second Approach: Quadratic Mapping Diagrams and Solving Quadratic Equations.

Review: Quadratic Example From Preface. ☺

$g(x) = 2\,(x-1)^2 + 3$

Steps for *g*:

1. **Linear:**

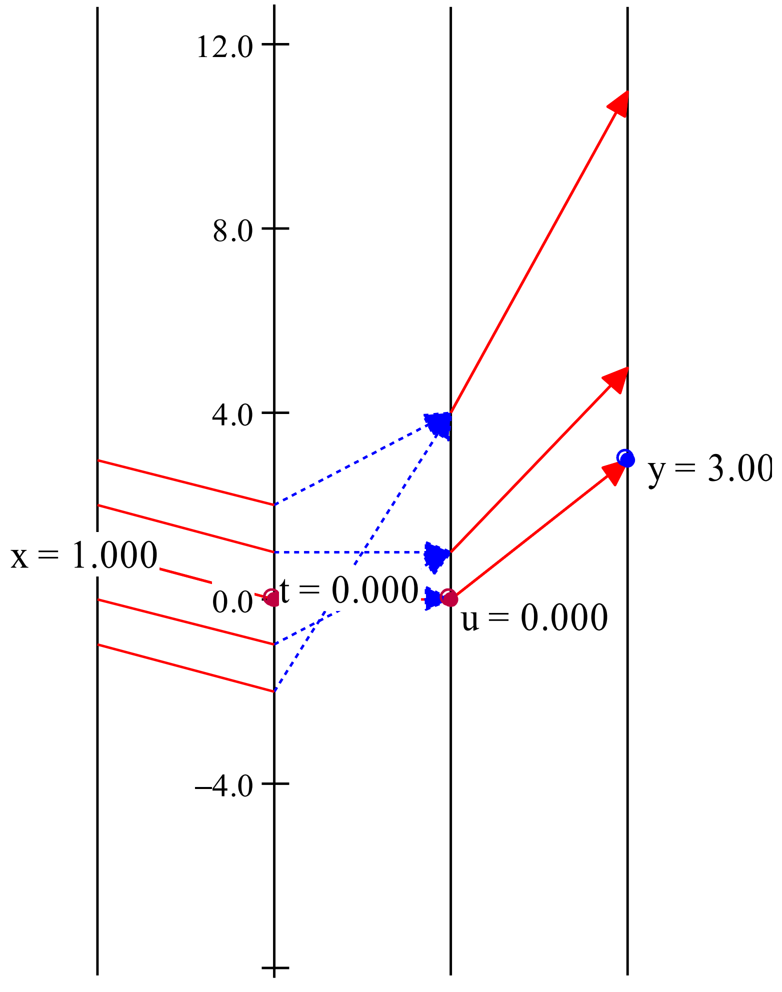
   Subtract 1.

2. **Square result.**

3. **Linear:**

   Multiply by 2

   then add 3.



Quadratic Functions

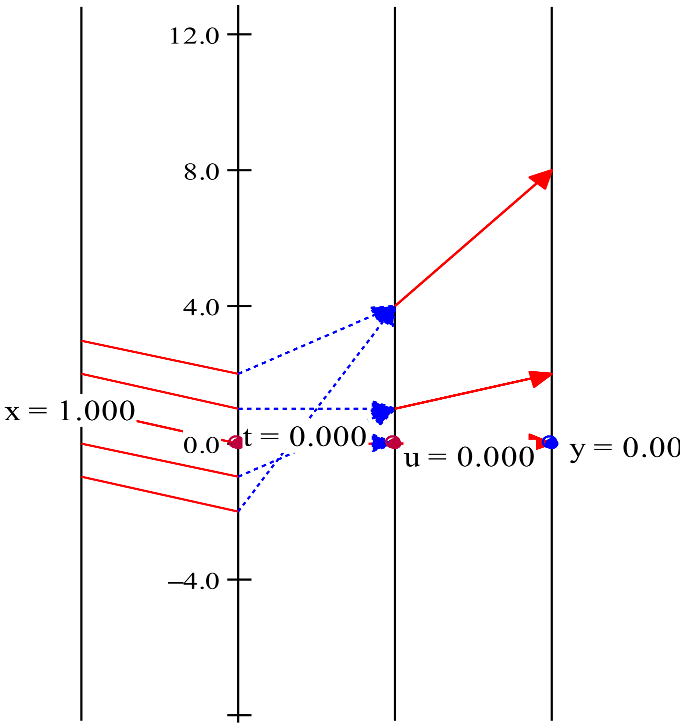
- Usually considered as a key example of the power of analytic geometry- the merger of algebra with geometry.
- The algebra of this study focuses on two distinct representations of these functions which mapping diagrams can visualize effectively to illuminate key features.
  - $f(x) = Ax^2 + Bx + C$
  - $f(x) = A(x-h)^2 + k$

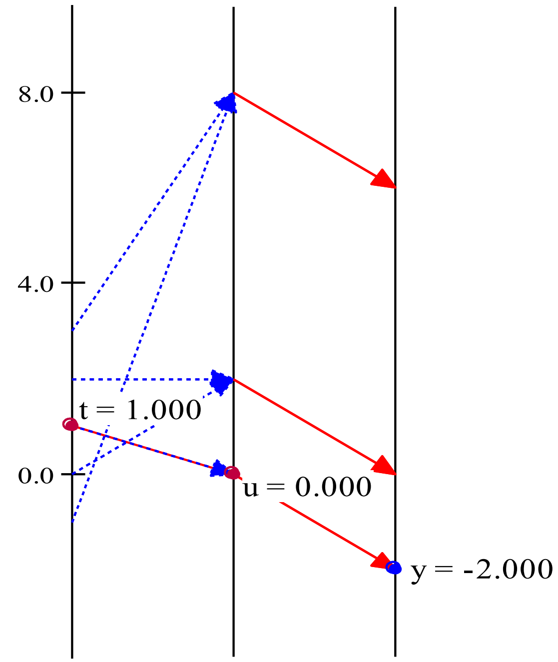
Examples

- Use compositions to visualize
  - $f(x) = 2(x-1)^2 = 2x^2 - 4x + 2$
  - $g(x) = 2(x-1)^2 + 3 = 2x^2 - 4x + 5$
- **Observe how even symmetry is transformed.**
- **These examples illustrate how a mapping diagram visualization of composition with linear functions can assist in understanding other functions.**

Quadratic Mapping diagrams

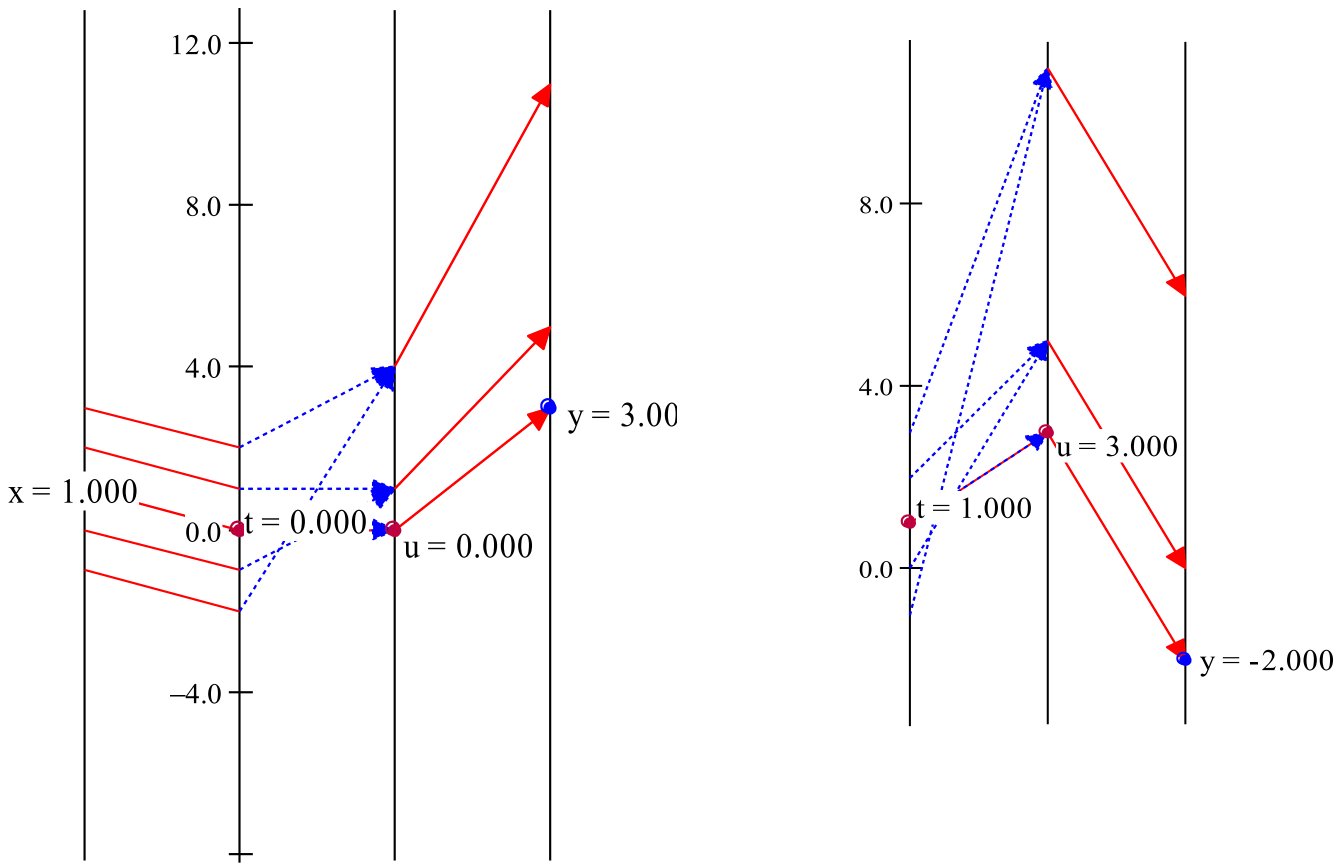
$f(x) = 2\,(x-1)^2 = 2x^2 - 4x + 2$





Quadratic Mapping diagrams

$$g(x) = 2\,(x-1)^2 + 3 \qquad = 2x^2 - 4x + 5$$



Quadratic Equations and Mapping diagrams

- Solve $f(x) = Ax^2 + Bx + C = 0$.
- Plan: Find 0 on the target axis, then trace back on any and all arrows that "hit" 0.
- Question: How does this connect to $x = -B/(2A)$ for symmetry and the issue of the number of solutions?

Think about These Problems

M.1 How would you use the Linear Focus to **find the mapping diagram for the function inverse for a linear function when m≠0?**

M.2 How does the **choice of axis scales** affect the **position of the linear function focus point** and its use in solving equations?

M.3 Describe the visual features of the mapping diagram for the quadratic function $f(x) = x^2$.
How does this generalize for *even* functions where $f(-x) = f(x)$?

M.4 Describe the visual features of the mapping diagram for the cubic function $f(x) = x^3$.
How does this generalize for *odd* functions where $f(-x) = -f(x)$?

MoreThink about These Problems

L.1 **Describe the visual features of the mapping diagram for the quadratic function *f* (*x*) = x^2.**
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? "Infinity"?

L.2 Describe the visual features of the mapping diagram for the quadratic function
$f(x) = A(x-h)^2 + k$ using composition with simple linear functions.
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? "Infinity"?

L.3 **Describe the visual features of a mapping diagram for the square root function $g(x) = \sqrt{x}$ and relate them to those of the quadratic $f(x) = x^2$.**
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? "Infinity"?

L.4 **Describe the visual features of the mapping diagram for the reciprocal function**
$f(x) = 1/x$.
Domain? Range? "Asymptotes" and "infinity"? Function Inverse?

L.5 Describe the visual features of the mapping diagram for the linear fractional function
$f(x) = A/(x-h) + k$ using composition with simple linear functions.
Domain? Range? "Asymptotes" and "infinity"? Function Inverse?

Thanks
The End!

☺

Questions?

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