Math 240 Proof Evaluation #6 Fall, 2006
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Proposition 1: log(2)/log(3) is irrational.
Proof: Assume that log(2)/log(3) = $\log_3(2)$ = p/q, with p, q integers and nonzero (no need to assume that the fraction is in lowest terms!) Then $2 = 3^{\frac{p}{q}}$ or $2^q = 3^p$.

But this is impossible, for the left hand side is even and the right hand side is odd.
EOP.

Proposition 2. $\sqrt{2}$ is irrrational.
Proof: (attributed to Dedekind) Suppose $\sqrt{2}$ - being positive - is a rational number. Then there is a positive integer *n* for which $n \times \sqrt{2}$ is an integer.
But this means that *n*($\sqrt{2}$ - integerpart($\sqrt{2}$)) is a positive integer that is smaller than *n*. However, *n*($\sqrt{2}$ - integerpart($\sqrt{2}$))× $\sqrt{2}$ is an integer. If n were chosen to be the smallest such positive integer this would give a contradiction**.**
EOP.

Proposition 3: There is an irrational number *r* such that $r^{\sqrt{2}}$ is rational.
Proof: If $\sqrt{3}^{\sqrt{2}}$ is a rational number, then $\sqrt{3}$ is the desired example. If $\sqrt{3}^{\sqrt{2}}$ is an irrational number then notice that $(\sqrt{3}^{\sqrt{2}})^{\sqrt{2}} = 3$, which is a rational number.
Hence either $\sqrt{3}$ or $\sqrt{3}^{\sqrt{2}}$ is an irrational number *r* such that $r^{\sqrt{2}}$ is rational.
EOP

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1. Consider the following statement:
   If a and b are relatively prime integers, then log(a)/log(b) is irrational.
   a. Discuss briefly how proposition 1 is a special case of this statement.
   b. Following the proof of proposition 1, prove the statement.

2. Using the argument of the proof of proposition 2 as a model, prove $\sqrt{3}$ is irrational.

3. In the proof of Proposition 3, the number $\sqrt{3}$ played an important role. Give a proof for the proposition without using the number $\sqrt{3}$. [Hint: Use $\sqrt{5}$.]

4. Assume that $\sqrt[3]{5}$ is an irrational number. Prove: There exists an irrational number *r* such that $r^{\sqrt[3]{5}}$ is a rational number.