Using Mapping Diagrams to Understand Functions
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Link for materials

http://users.humboldt.edu/flashman/CMC2013/CMC3_LINKS.html

Background Questions

- Hands Up or Down…
  1. Are you familiar with Mapping Diagrams?
  2. Have you used Mapping Diagrams to teach functions?
  3. Have you used Mapping Diagrams to teach content besides function definitions?

Mapping Diagrams

A.k.a.

Function Diagrams

Dynagraphs

Preface: Quadratic Example
Will be reviewed at end. ☺

$g(x) = 2\,(x\text{-}1)^2 + 3$

Steps for *g*:

1. **Linear:**

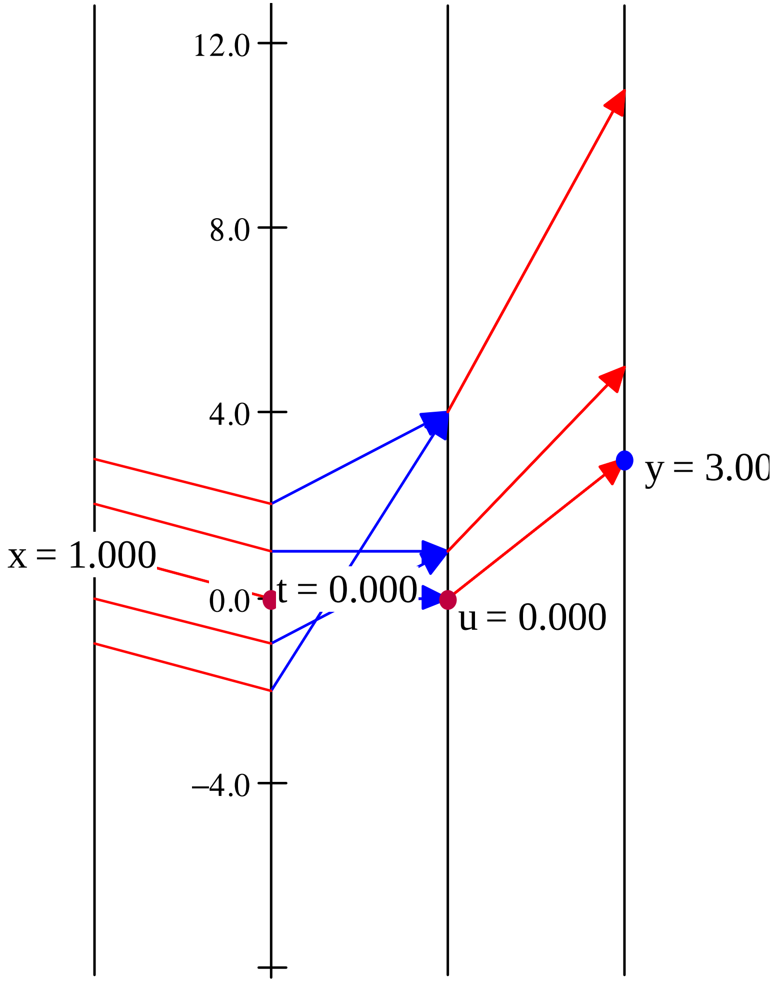
   Subtract 1.

2. **Square result.**

3. **Linear:**

   Multiply by 2

   then add 3.

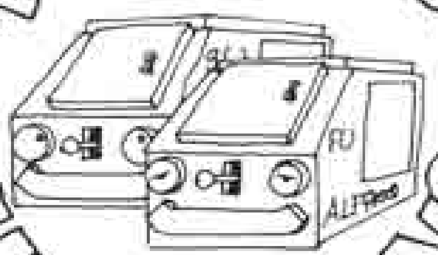


Written by Howard Swann and John Johnson

A fun source for visualizing functions with mapping diagrams at an elementary level.

Original version Part 1 (1971)
Part 2 and Combined (1975)

RULES FOR FUNCTIONS



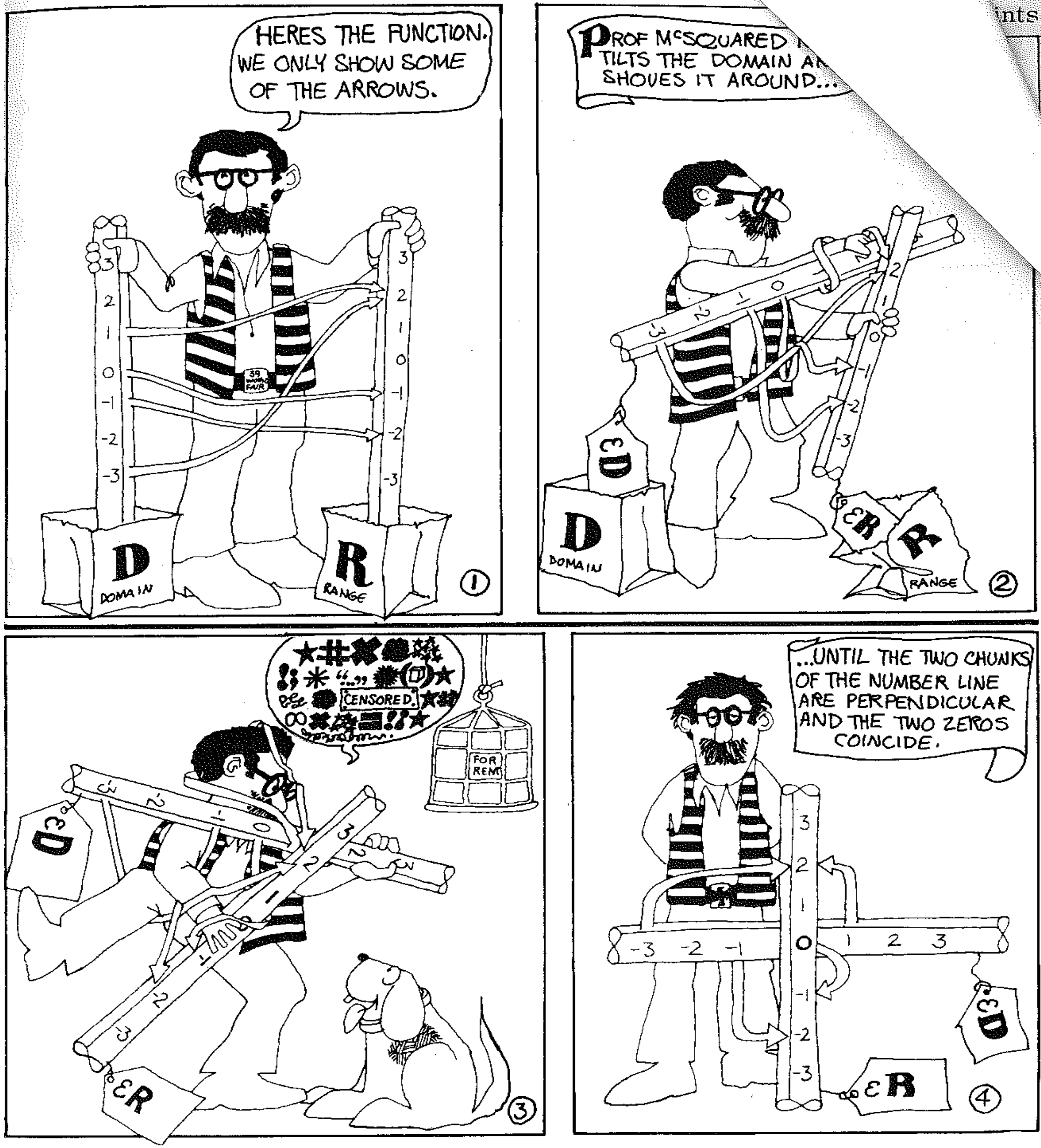
YOUR FRIENDLY NEIGHBORHOOD FUNCTION CONSISTS OF TWO SETS AND A BUNCH OF ARROWS THAT OBEY

RULE 1 THE ARROWS ALWAYS START FROM THE SAME SET, CALLED THE **DOMAIN** AND GO TO THE OTHER SET, CALLED THE **RANGE**.

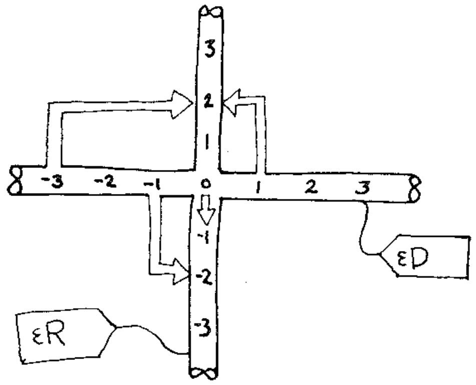
RULE 2 EVERYTHING IN THE DOMAIN-SET MUST HAVE EXACTLY ONE ARROW FROM IT. EVERYTHING IN THE RANGE-SET MUST HAVE AT LEAST ONE ARROW TO IT.
(IT'S OK, TO HAVE 2 ARROWS TO 1 THING.)

So two or more arrows can hit the same thing in the range-set, but only one arrow can come from any particular thing in the domain-set.

Using arrows in the RULES unfortunately has its drawbacks —as functions become more elaborate, the arrows can get pretty difficult to follow . . . .


HERES THE FUNCTION. WE ONLY SHOW SOME OF THE ARROWS.
D
DOMAIN
R
RANGE
PROF McSQUARED
TILTS THE DOMAIN A
SHOVES IT AROUND...
εD
D
DOMAIN
εR
R
RANGE
CENSORED
FOR RENT
εD
εR
...UNTIL THE TWO CHUNKS OF THE NUMBER LINE ARE PERPENDICULAR AND THE TWO ZEROS COINCIDE.
εD
εR

Fix up each arrow so it starts off straight up or down and then makes a right-angle turn directly over to the range.



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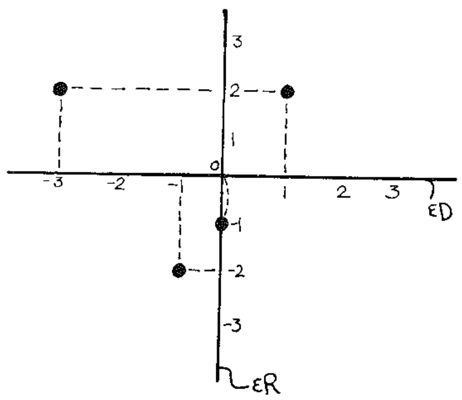
Now we can preserve ALL the information about the function by just keeping the DOTS where the function-arrows turn.



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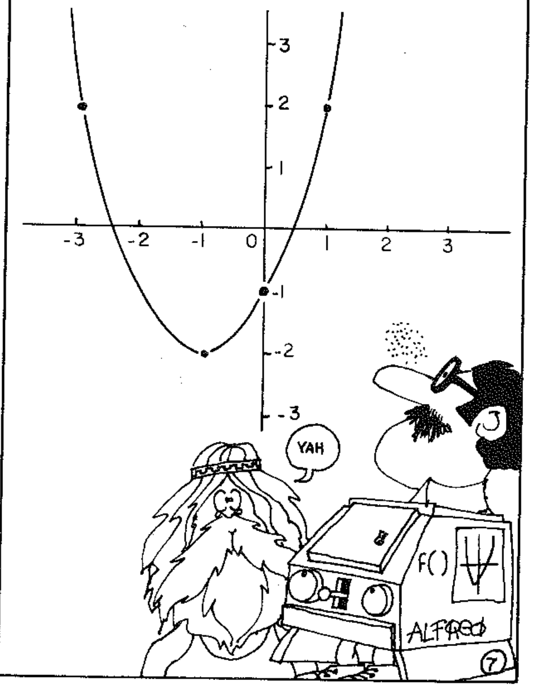
So the dots tell us all about the function. There are usually many more arrows than we have shown and thus more dots. In fact, usually there are so many that the dots make a solid curve. Showing how such a function works using just arrows from the domain-set to the range-set would really be a problem.



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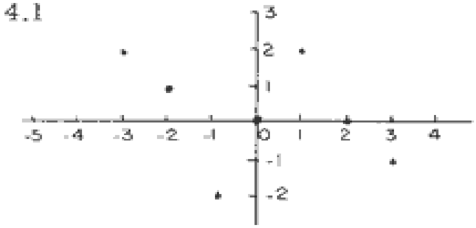
Remember that the DOMAIN of any function is always part of the horizontal ⟷ line, called the "x-axis" because any arbitrarily chosen thing in the domain-set is usually called "x." The RANGE is always part of the vertical (↕) line, called the "y - axis " because an arbitrary thing in the range-set is usually called "y."

EXERCISES

The answers to most of these exercises are really just approximate, since they depend on reading graphs.

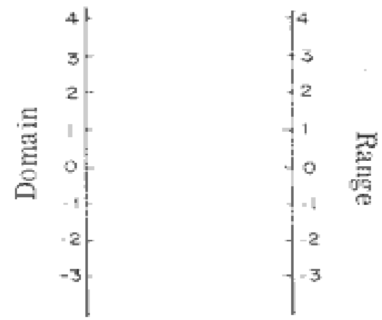
I.4.1



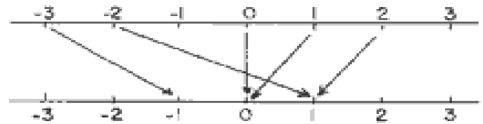
(a) Label the dots.
(b) If the dots represent a function, what is

f(3) = ?
f(-3) = ?
f(1) = ?
f(2) = ?
f(0) = ?

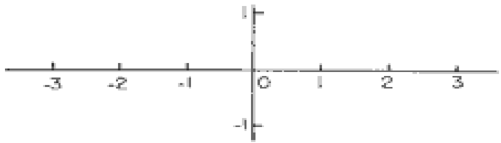
(c) Put in the appropriate arrows to show what corresponds to what in this function.



I.4.2 If f( ) is this correspondence;

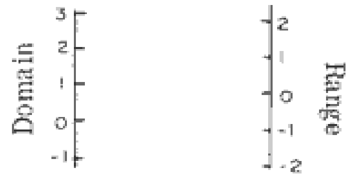


draw its graph and label the dots on the graph. (It will be just 5 dots.)



I.4.3 If the domain of f( ) is
$\{-1, 0, \frac{1}{2}, 2, 3\}$
and $f(-1) = -1$, $f(0) = 0$, $f(\frac{1}{2}) = 1$, $f(2) = -2$ and $f(3) = -2$,

(a) Put in the appropriate arrows to show what corresponds to what in this function.



(b) Draw the graph of f( ). (It will just be five dots.)

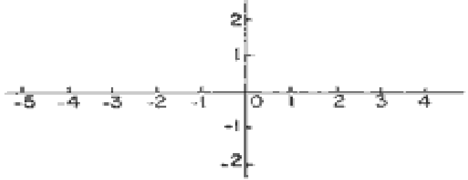
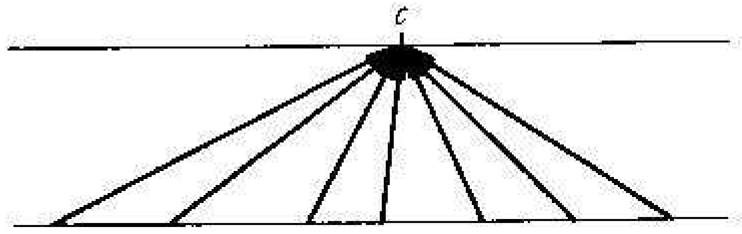
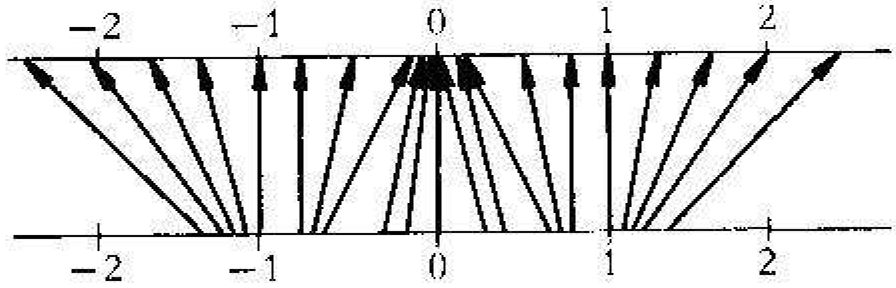


Figure from Ch. 5
Calculus by M. Spivak



(a) $f(x) = c$



(b) $f(x) = x^3$

FIGURE 2

Main Resource for Remainder of Webinar

- Mapping Diagrams from A (algebra) B(asics) to C(alculus) and D(ifferential) E(quation)s. A Reference and Resource Book on Function Visualizations Using Mapping Diagrams (Preliminary Sections- NOT YET FOR publication)

- http://users.humboldt.edu/flashman/MD/section-1.1VF.html

Linear Mapping diagrams

We begin our more detailed introduction to mapping diagrams by a consideration of linear functions :

$$\text{“ } y = f(x) = mx + b \text{ ”}$$

Visualizing Linear Functions

- Linear functions are both necessary, and understandable- even without considering their graphs.
- There is a sensible way to visualize them using "mapping diagrams."
- Examples of <u>important function features (like slope and intercepts)</u> can be illustrated with mapping diagrams.
- Activities for students engage understanding both function and linearity concepts.
- Mapping diagrams use simple straight edges as well as technology GeoGebra and SAGE.

Linear Functions: Tables

x	5 x - 7
3	
2	
1	
0	
-1	
-2	
-3	

Complete the table.

$x = 3,2,1,0,-1,-2,-3$

$f(x) = 5x - 7$

$f(0) = \_\_\_\_?$

For which x is $f(x) > 0$?

Linear Functions: Tables

X	5 x – 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Complete the table.

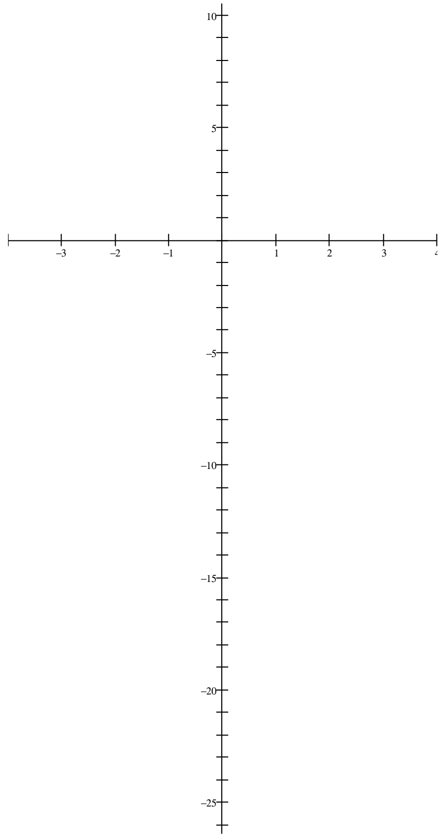
$x = 3,2,1,0,-1,-2,-3$

$f(x) = 5x - 7$

$f(0) =$ ____?

For which x is $f(x) > 0$?

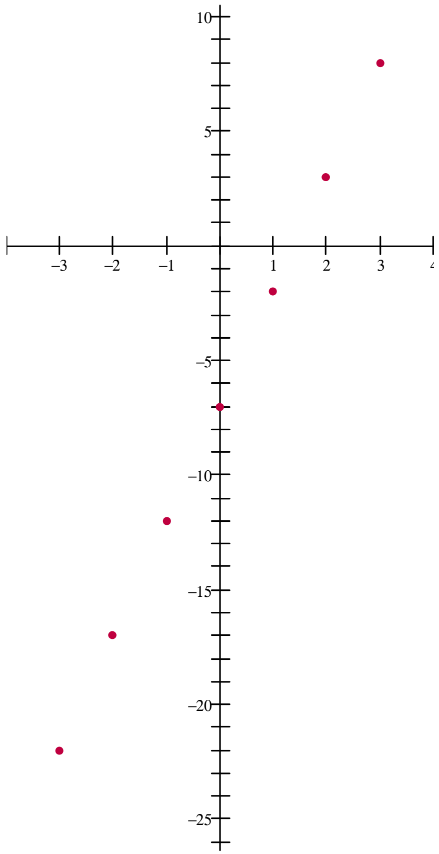
Linear Functions: On Graph



Plot Points (x , 5x - 7):

X	5 x – 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

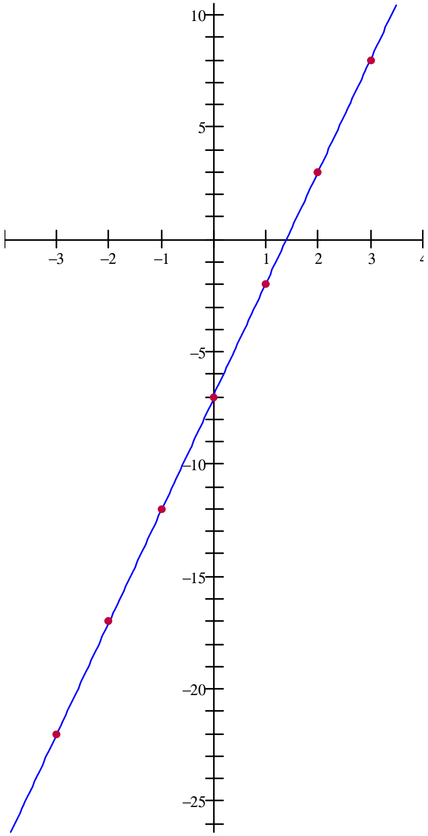
Linear Functions: On Graph



Connect Points
(x , 5x - 7):

X	5 x - 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Linear Functions: On Graph



Connect the Points

X	5 x – 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Linear Functions: Mapping diagrams
What happens before the graph.

- Connect point x to point 5x – 7 on axes

X	5 x – 7
3	8
2	3
1	-2
0	-7
-1	-12
-2	-17
-3	-22

Linear Functions: Mapping diagrams
What happens before the graph.

X	5 x - 7
3	8
2	3
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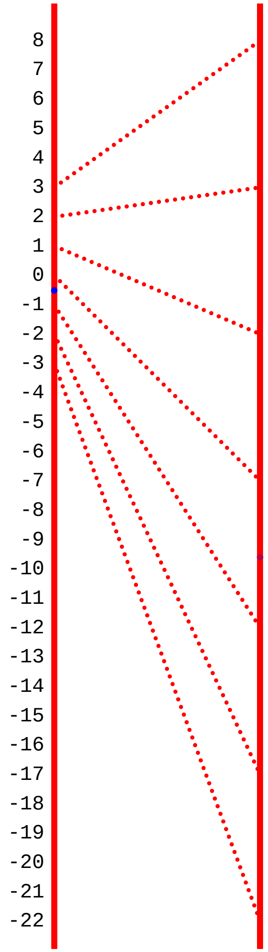
-3	-22



Try It…

Do first problem on the worksheet now:

Worksheet.VF1.pdf.

Thumbs up when you are ready to proceed.

Examples on Excel / Geogebra / SAGE

- Excel example
- Geogebra example

Do worksheet problem #2 (group of 3 or 4?)

Worksheet.LF.pdf.

Thumbs up when you are ready to proceed.

Simple Examples are important!

- $f(x) = x + C$ Added value: C
- $f(x) = mx$ Scalar Multiple: m
  Interpretations of m:
  - slope
  - rate
  - Magnification factor
  - $m > 0$: Increasing function
  - $m = 0$: Constant function [WS Example]
  - $m < 0$: Decreasing function [WS Example]

Simple Examples are important!

$f(x) = mx + b$ with a mapping diagram -- Five examples:

- Example 1: $m = -2$; $b = 1$: $f(x) = -2x + 1$
- Example 2: $m = 2$; $b = 1$: $f(x) = 2x + 1$
- Example 3: $m = \frac{1}{2}$; $b = 1$: $f(x) = \frac{1}{2}x + 1$
- Example 4: $m = 0$; $b = 1$: $f(x) = 0x + 1$
- Example 5: $m = 1$; $b = 1$: $f(x) = x + 1$

Which diagram(s) have crossing arrows?

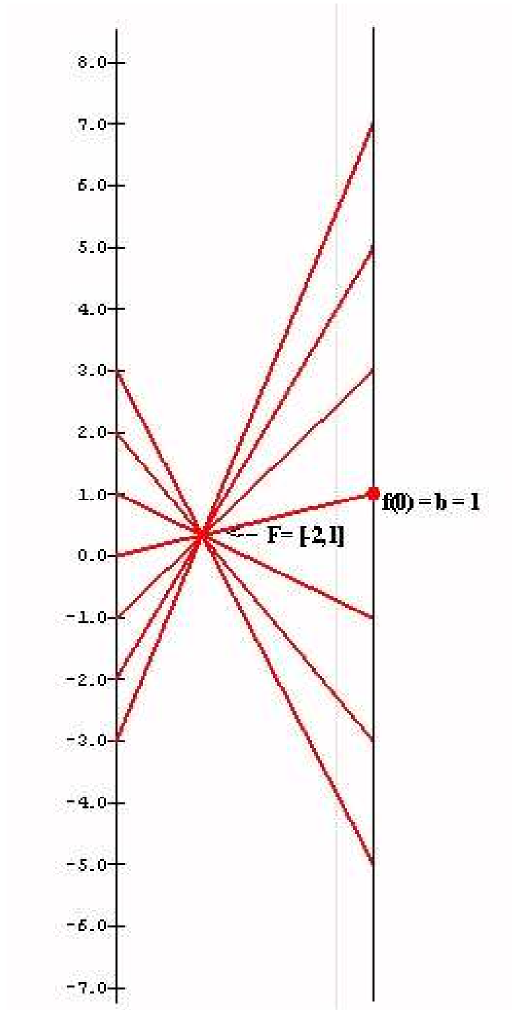
Visualizing $f(x) = \boldsymbol{m}x + \boldsymbol{b}$ with a mapping diagram -- Five examples:

Example 1: m = -2; b = 1

$f(x) = \mathbf{-2}x + \mathbf{1}$

- Each arrow passes through a single point, which is labeled **F = [- 2,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $-2x + 1$,

    which corresponds to the linear function's value for the point/number, *x*.



Visualizing $f(x) = \boldsymbol{m}x + \boldsymbol{b}$ with a mapping diagram -- Five examples:

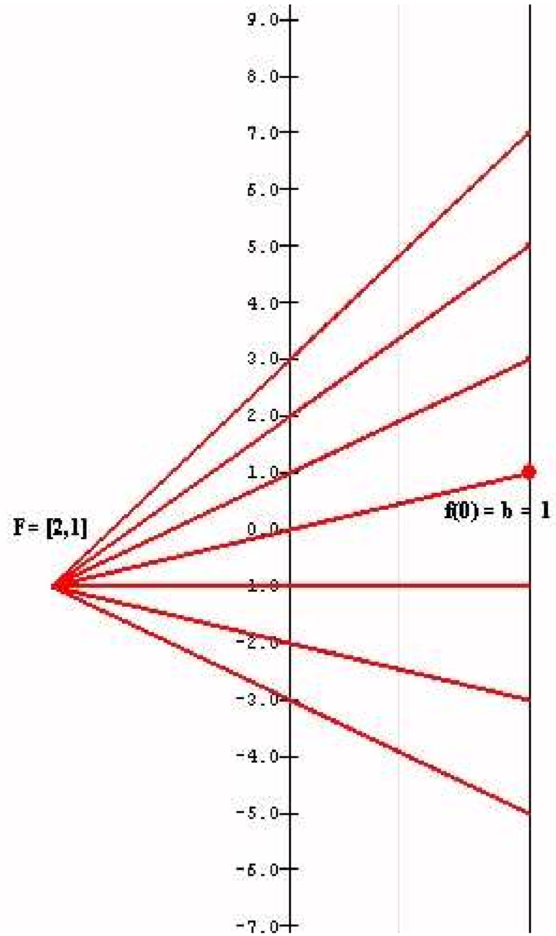
Example 2: $m = 2$; $b = 1$
$f(x) = 2x + 1$

Each arrow passes through a single point, which is labeled

F = [2,1].

- The point **F** completely determines the function *f*.
  - **given a point** / number, *x*, on the source line,
  - there is a **unique arrow passing through F**
  - **meeting** the target line at **a unique point** / number, $2x + 1$,

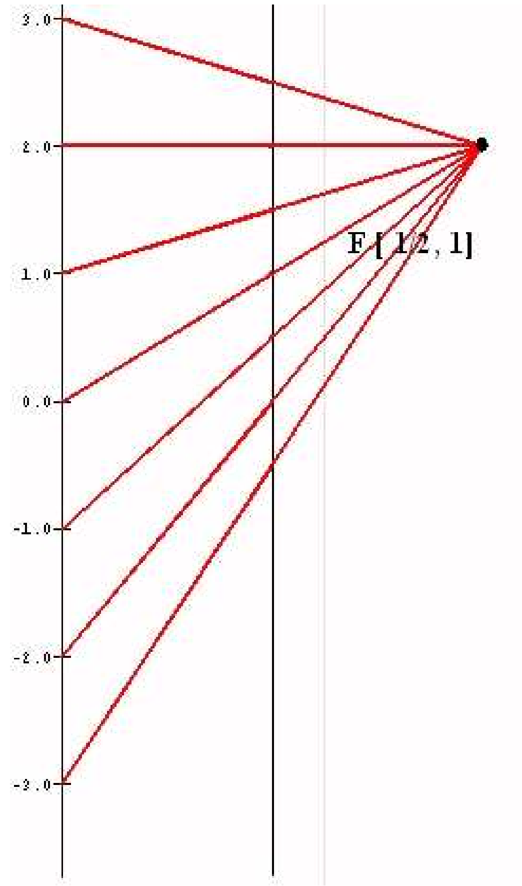
which corresponds to the linear function's value for the point/number, *x*.



Visualizing *f* (*x*) = ***mx*** + ***b*** with a mapping diagram -- Five examples:

- **Example 3: *m* = 1/2; *b* = 1**
  f(x) = ½ x + 1
- Each arrow passes through a single point, which is labeled **F = [1/2,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, *x*, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, ½ *x* + 1,

    which corresponds to the linear function's value for the point/number, *x*.

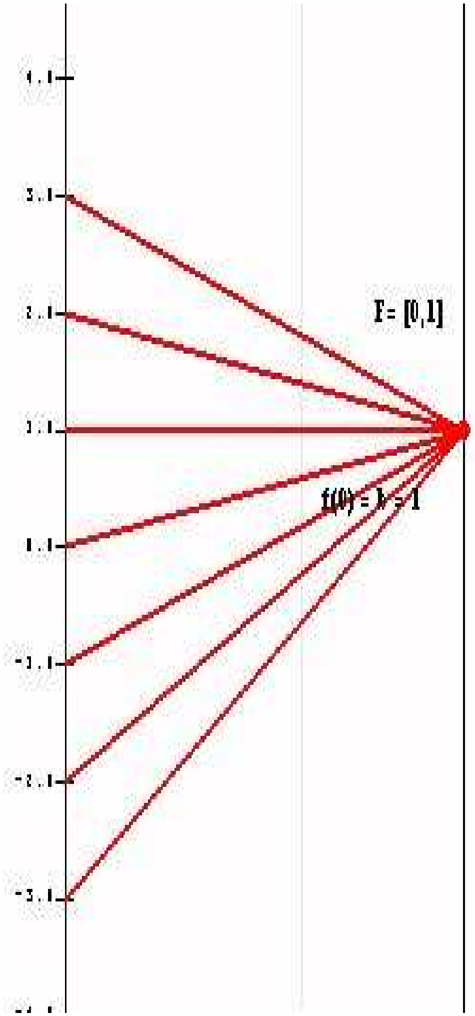


Visualizing $f(x) = \boldsymbol{mx} + \boldsymbol{b}$ with a mapping diagram -- Five examples:

- **Example 4: $m = 0$; $b = 1$**
  $f(x) = 0\ x + 1$

- Each arrow passes through a single point, which is labeled **F = [0,1]**.
  - The point **F** completely determines the function *f*.
    - **given a point** / number, x, on the source line,
    - there is a **unique arrow passing through F**
    - **meeting** the target line at **a unique point** / number, $f(x)=1$,

    which corresponds to the linear function's value for the point/number, x.



Visualizing $f(x) = \boldsymbol{mx} + \boldsymbol{b}$ with a mapping diagram -- Five examples

Example 5: $\boldsymbol{m} = 1;\ \boldsymbol{b} = 1$

$f(x) = x + 1$

- Unlike the previous examples, in this case it is not a single point that determines the mapping diagram, but the single arrow from 0 to 1, which we designate as F[1,1]
- It can also be shown that this single arrow completely determines the function. Thus, given a point / number, x, on the source line, there is a unique arrow passing through ***x parallel to*** F[1,1] meeting the target line a unique point / number, $x + 1$, which corresponds to the linear function's value for the point/number, x.
  - The single arrow completely determines the function f.
    - **given a point** / number, x, on the source line,
    - there is a **unique arrow** through ***x parallel to*** F[1,1]
    - **meeting** the target line at **a unique point** / number, $x + 1$,

which corresponds to the linear function's value for the point/number, x.

Function-Equation Questions

with linear focus points

- Solve a linear equations:

  $2x+1 = 5$

  – Use focus to find x.

More on Linear Mapping diagrams

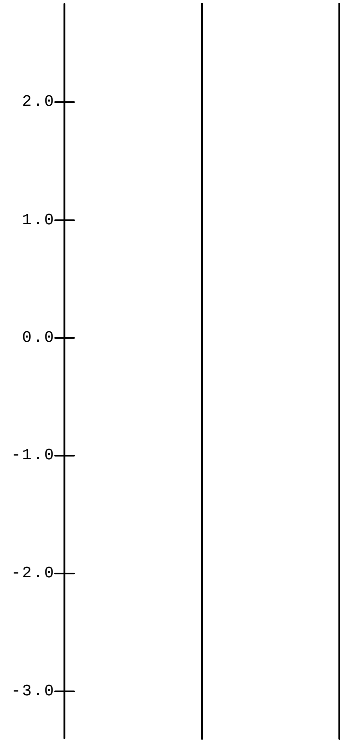
We continue our introduction to mapping diagrams by a consideration of the **composition of linear functions**.

Compositions are keys!

An example of composition with mapping diagrams of simpler (linear) functions.

- $g(x) = 2x$; $h(u)=u+1$
- $f(x) = h(g(x)) = h(u)$ where $u = g(x) = 2x$
- $f(x) = (2x) + 1 = 2x + 1$

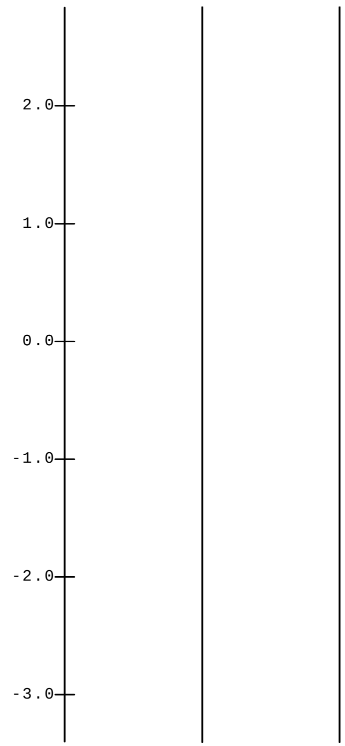
  $f(0) = 1$ magnification $= 2$



Compositions are keys!

Linear Functions can be understood and visualized as compositions with mapping diagrams of simpler linear functions.

- $f(x) = 2x + 1 = (2x) + 1$:
  - $g(x) = 2x$; $h(u)=u+1$
  - $f(0) = 1$ magnification = 2



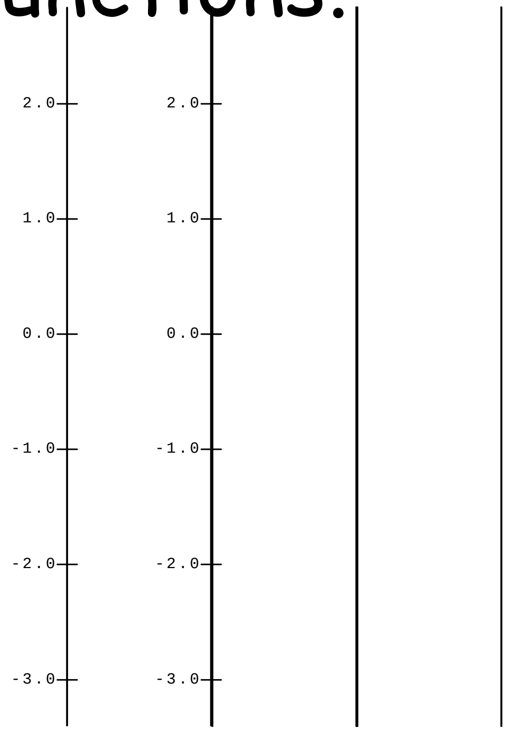
Compositions are keys!

Linear Functions can be understood and visualized as compositions with mapping diagrams of simpler linear functions.

Example: $f(x) = 2(x-1) + 3$

$g(x)=x-1 \quad h(u)=2u; \ k(t)=t+3$

- $f(1)= 3$ magnification = 2



Question for Thought

- For which functions would mapping diagrams add to the understanding of composition?
- In what other contexts are composition with “x+h” relevant for understanding function identities?
- In what other contexts are composition with “-x” relevant for understanding function identities?

Inverses, Equations and Mapping diagrams

- Inverse: If $f(x) = y$ then $invf(y)=x$.
- So to find $invf(b)$ we need to find any and all x that solve the equation $f(x) = b$.
- How is this visualized on a mapping diagram?
- Find b on the target axis, then trace back on any and all arrows that "hit"b.

Mapping diagrams and Inverses

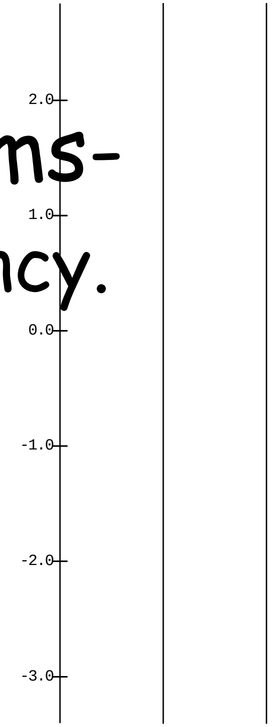
Inverse linear functions:

[In class approach]

- Use transparency for mapping diagrams-
  - Copy mapping diagram of f to transparency.
  - Flip the transparency to see mapping diagram of inverse function g.

    ("before or after")

    invg(g(a)) = a;   g(invg(b)) = b;



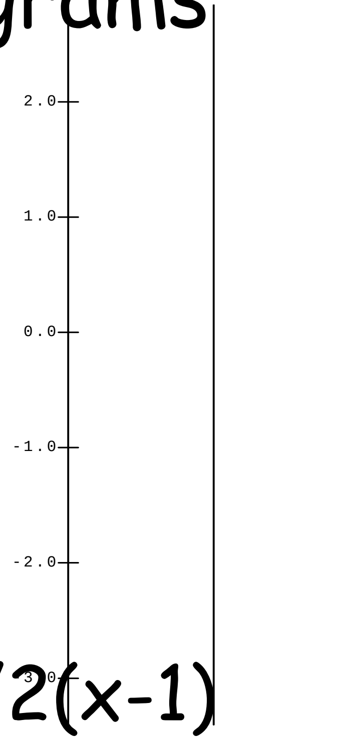
- Example i: $g(x) = 2x$; $invg(x) = 1/2\ x$
- Example ii: $h(x) = x + 1$; $invh(x) = x - 1$

Mapping diagrams and Inverses

Inverse linear functions:

- socks and shoes with mapping diagrams
- $g(x) = 2x$; $invf(x) = 1/2\ x$
- $h(x) = x + 1$; $invh(x) = x - 1$

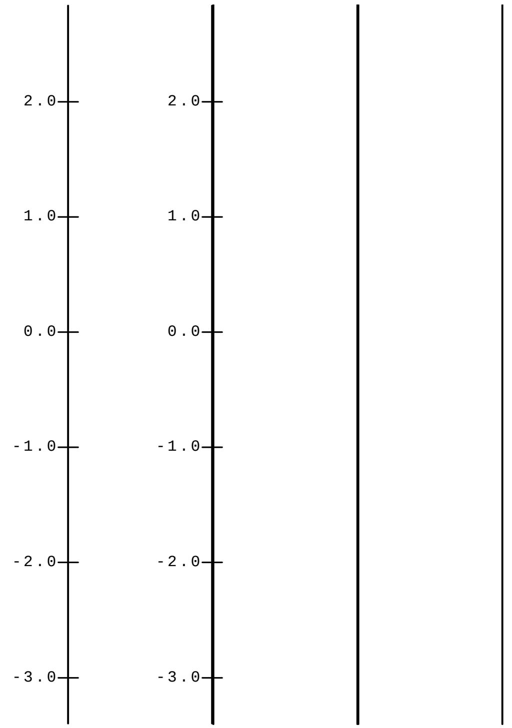
- $f(x) = 2\ x + 1 = (2x) + 1$
  - $g(x) = 2x$; $h(u)=u+1$
  - inverse of f: $invf(x)=invh(invg(x))=1/2(x-1)$



Mapping diagrams and Inverses

Inverse linear functions:

- “socks and shoes” with mapping diagrams
- f(x) = 2(x-1) + 3:
  - g(x)=x-1 h(u)=2u; k(t)=t+3
  - Inverse of f: 1/2(x-3) +1



Question for Thought

- For which functions would mapping diagrams add to the understanding of inverse functions?
- How does “socks and shoes” connect with solving equations and justifying identities?

Other Topics for Mapping Diagrams Before Calculus:

- Quadratic Functions
- Exponential and Logarithmic Functions
- Trigonometric Functions

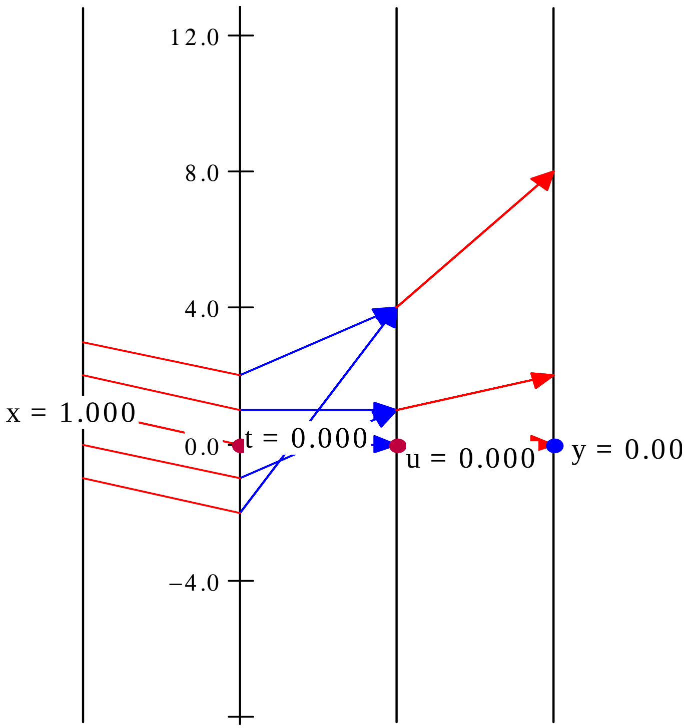
Examples

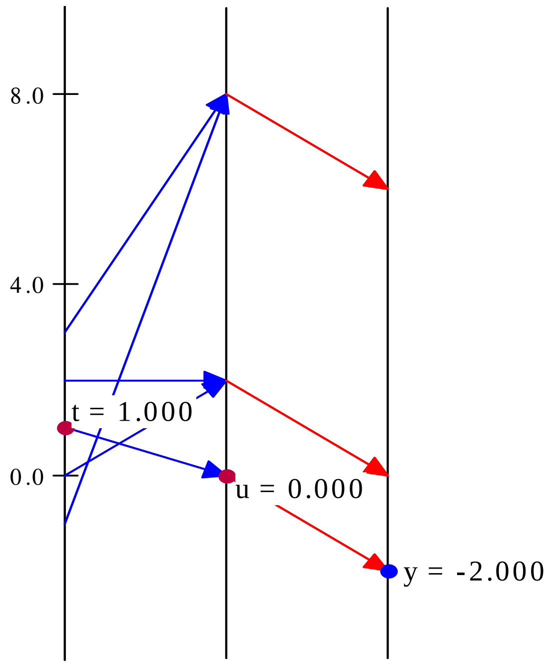
- Use compositions to visualize
  - $f(x) = 2\,(x-1)^2 = 2x^2 - 4x + 2$
  - $g(x) = 2\,(x-1)^2 + 3 = 2x^2 - 4x + 5$

- **Observe how even symmetry is transformed.**
- **These examples illustrate how a mapping diagram visualization of composition with linear functions can assist in understanding other functions.**

Quadratic Mapping diagrams

$$f(x) = 2\,(x-1)^2 \qquad = 2x^2 - 4x + 2$$





Quadratic Example From Preface. ☺

$g(x) = 2\,(x\text{-}1)^2 + 3$

Steps for *g*:

1. **Linear:**

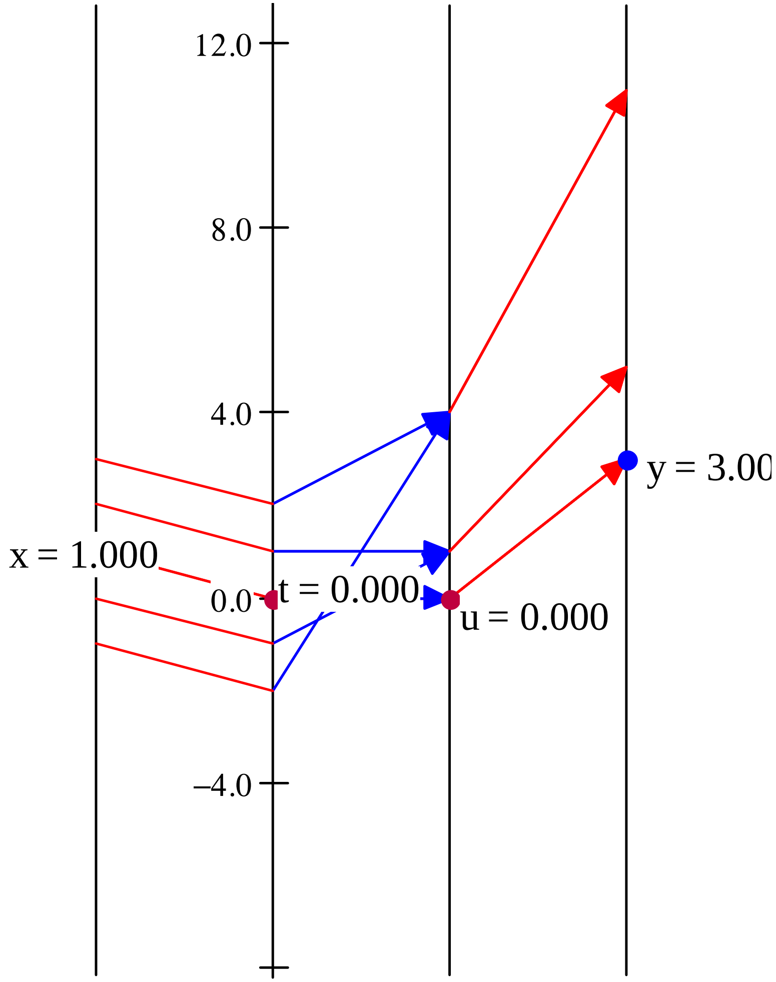
   Subtract 1.

2. **Square result.**

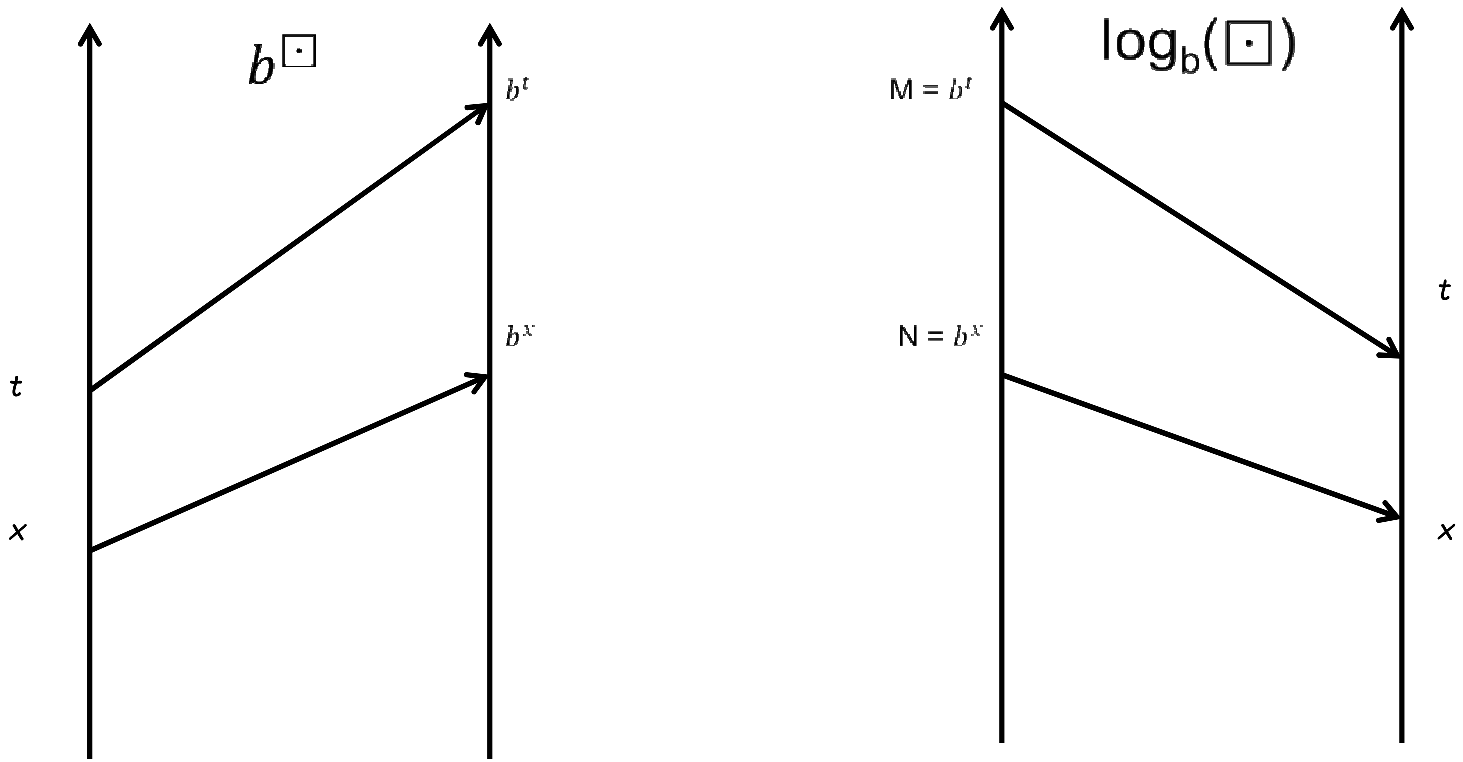
3. **Linear:**

   Multiply by 2

   then add 3.

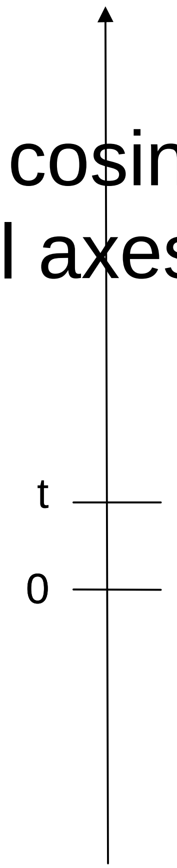


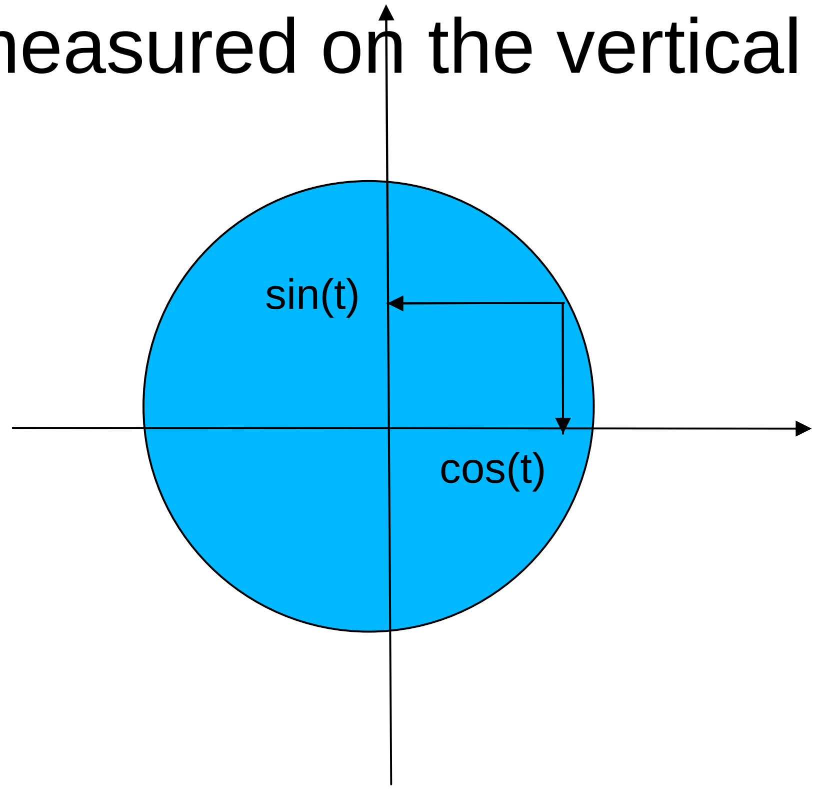
Mapping diagrams for exponential functions and "inverse"



Seeing the functions on the unit circle with mapping diagram.

Sine and cosine of t measured on the vertical and horizontal axes.

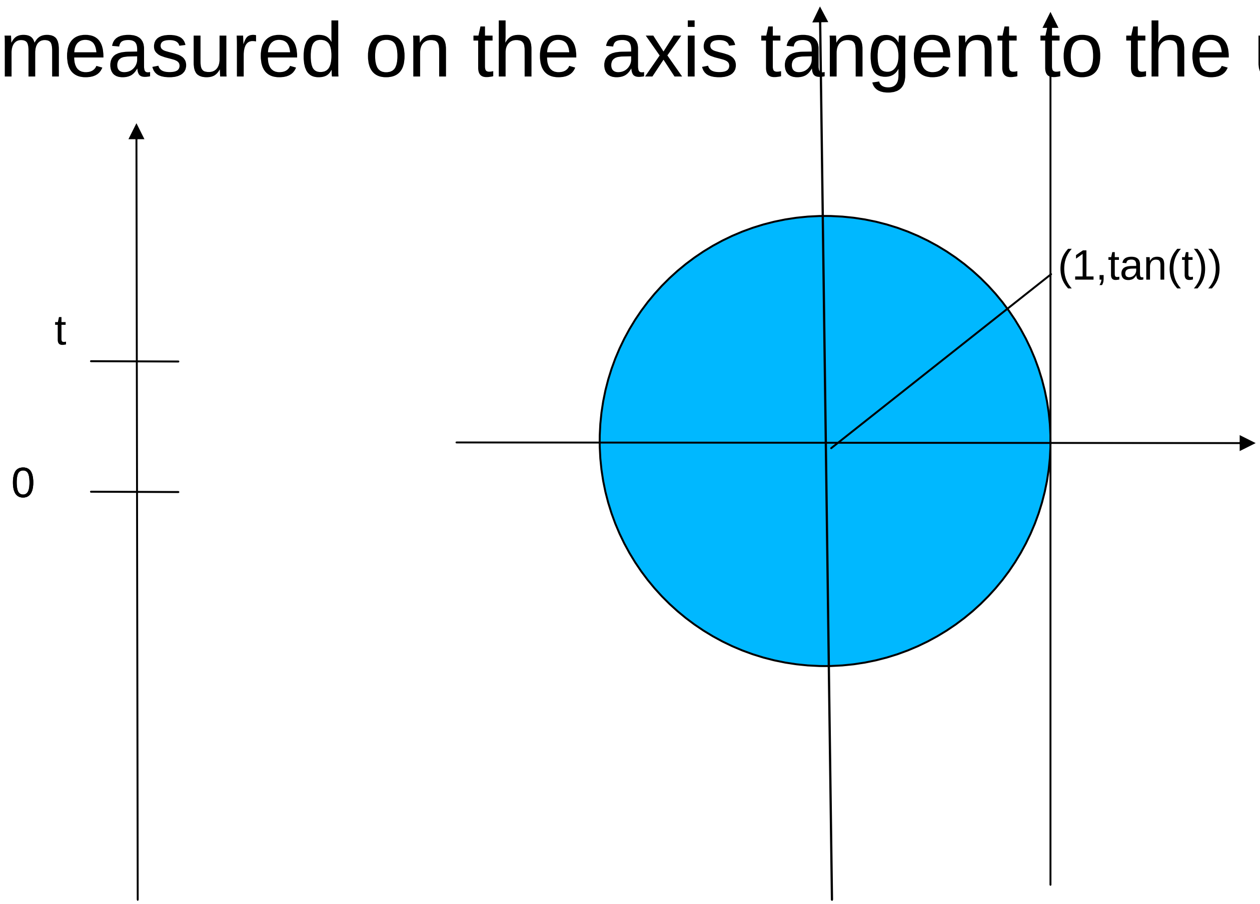




Note the visualization of periodicity.

Tangent Interpreted on Unit Circle

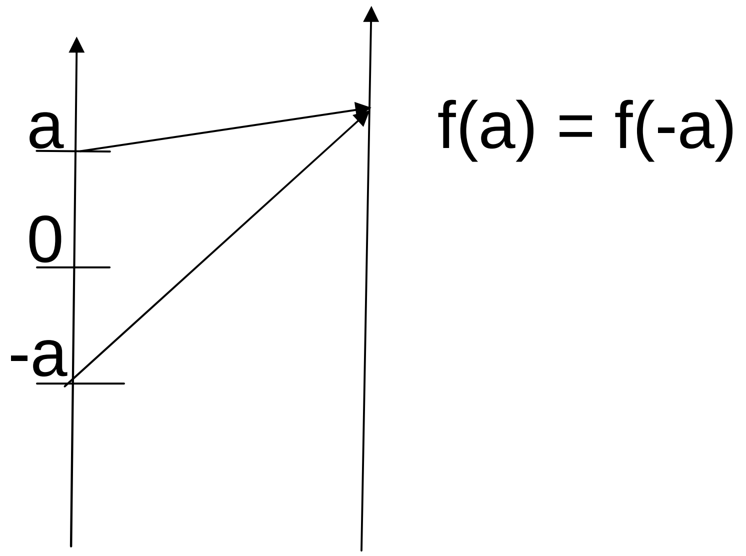
- Tan(t) measured on the axis tangent to the unit circle.



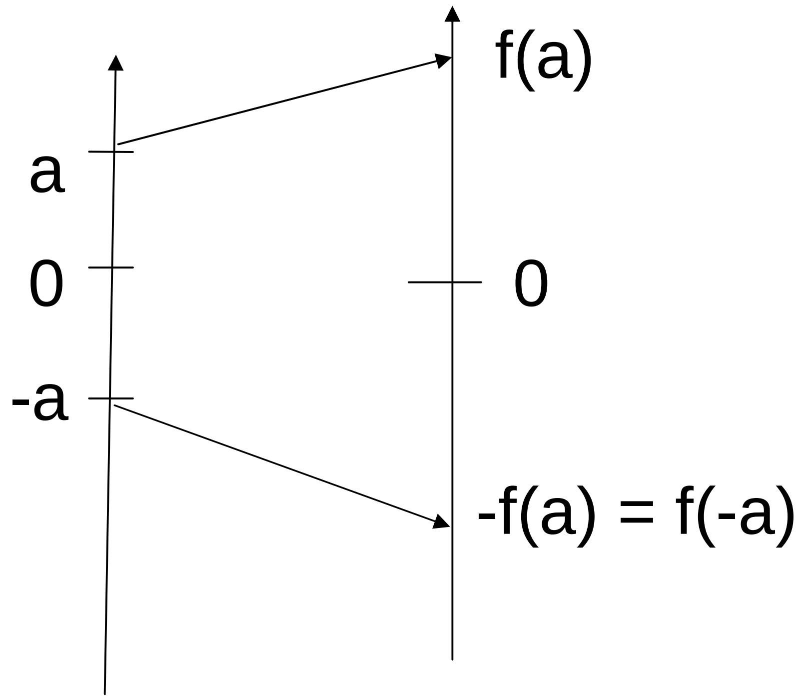
- Note the visualization of periodicity.

Even and odd on Mapping diagrams

Even

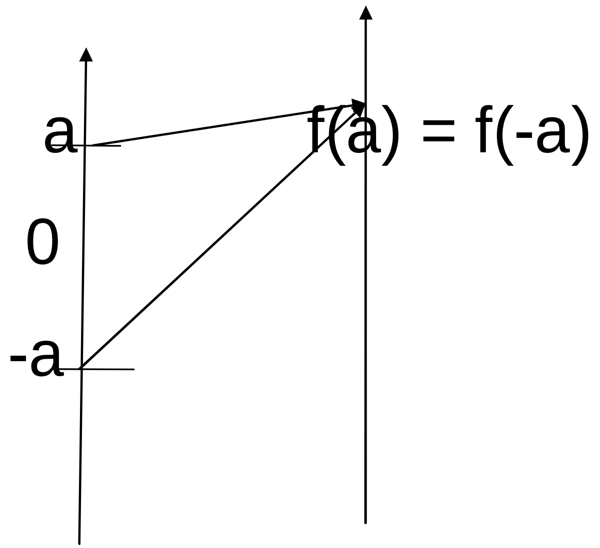


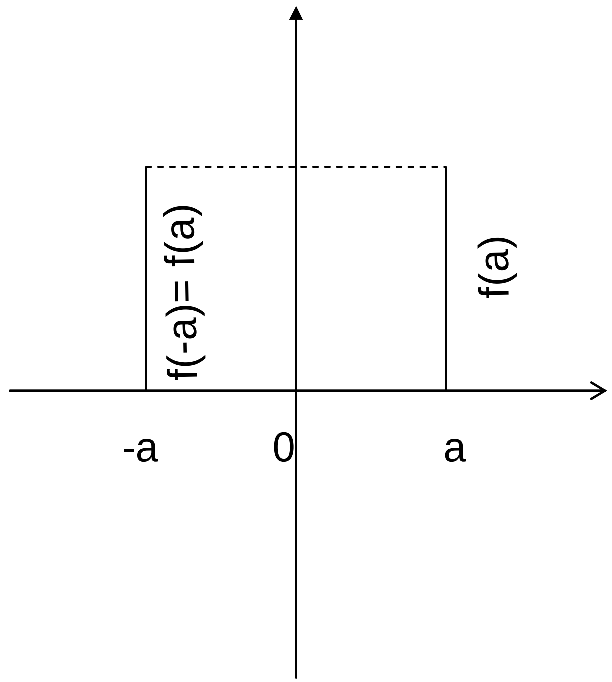
Odd



Even Function Mapping Figures and Graphs

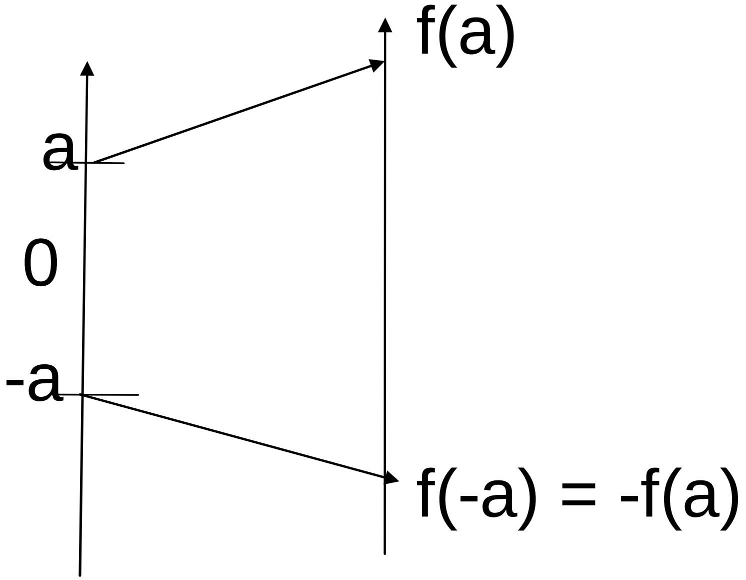
An Even Function

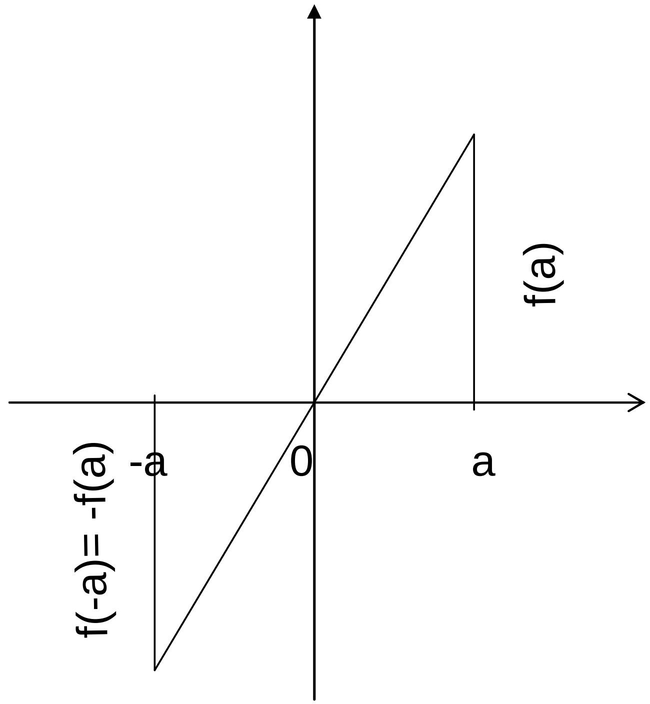




Odd Function Mapping Figures and Graphs

An Odd Function



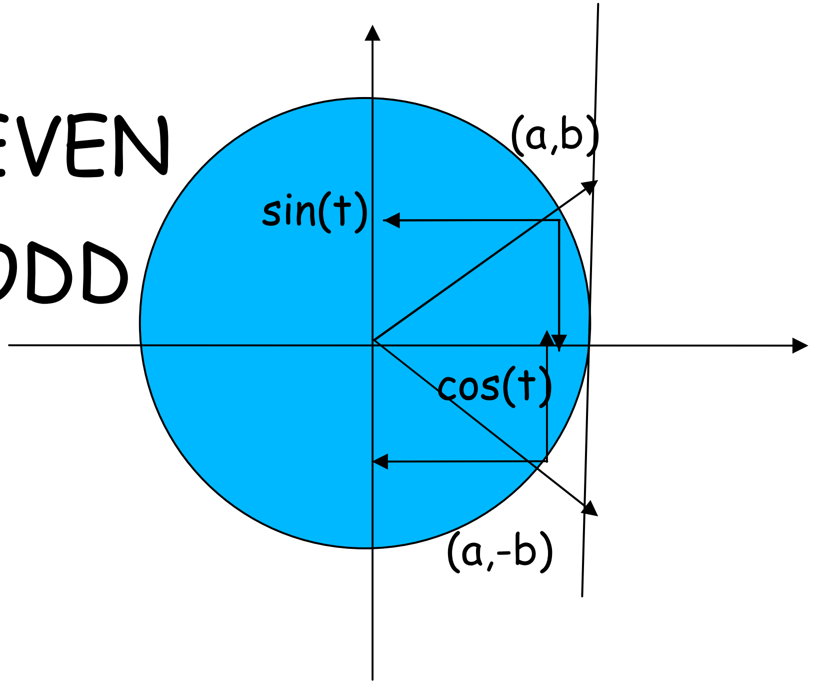


Trigonometric functions and symmetry:

cos(-t) = cos(t) for all t. EVEN

sin(-t) = -sin(t) for all t.   ODD

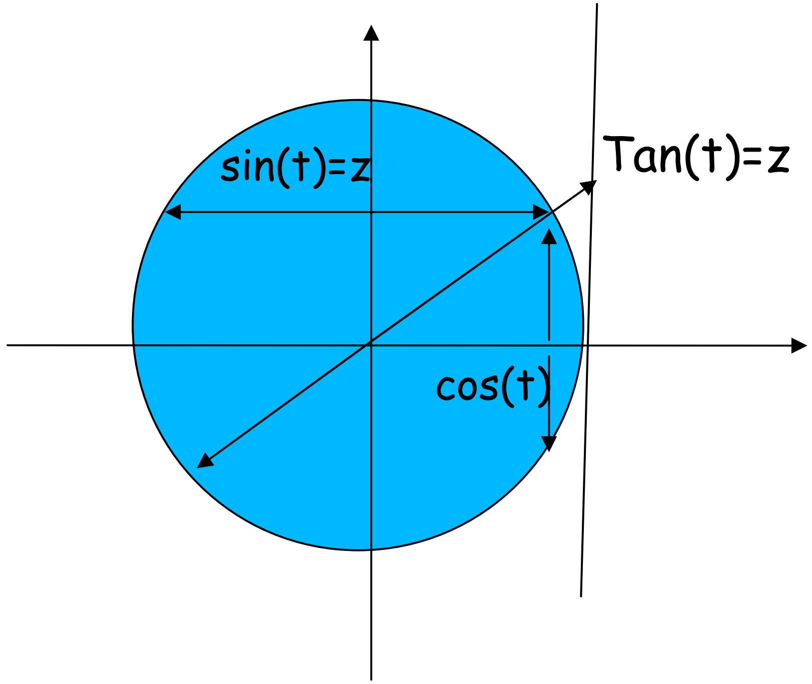
tan(-t) = - tan(t) for all t.



Justifications from unit circle mapping diagrams for sine, cosine and tangent..

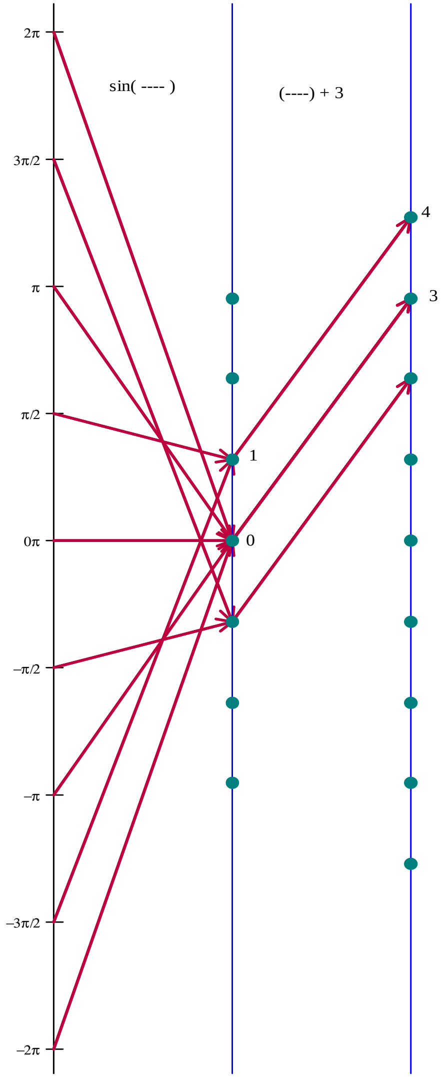
Solving Simple Trig Equations:

Solve trig(t)=z from unit circle mapping diagrams for sine, cosine and tangent.



Compositions with Trig Functions

Example: y = f(x) =sin(x) + 3



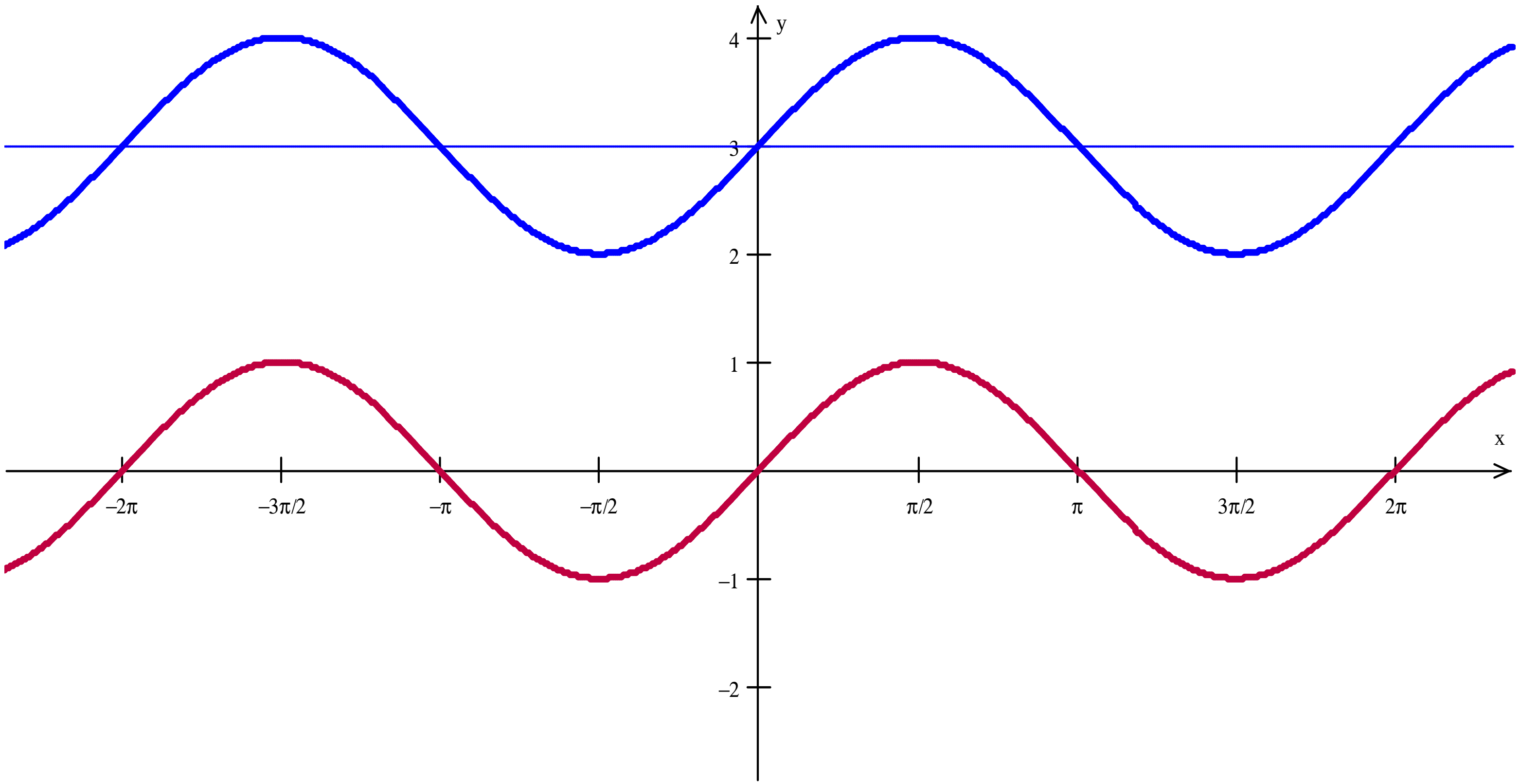
- Mapping diagram for
- y =f(x) = sin(x) + 3 considered as a composition:
- First: u = sin(x)
- Second: y =u+3 so the result is
- y = (sin(x)) + 3

Example: Graph of y = sin(x) + 3

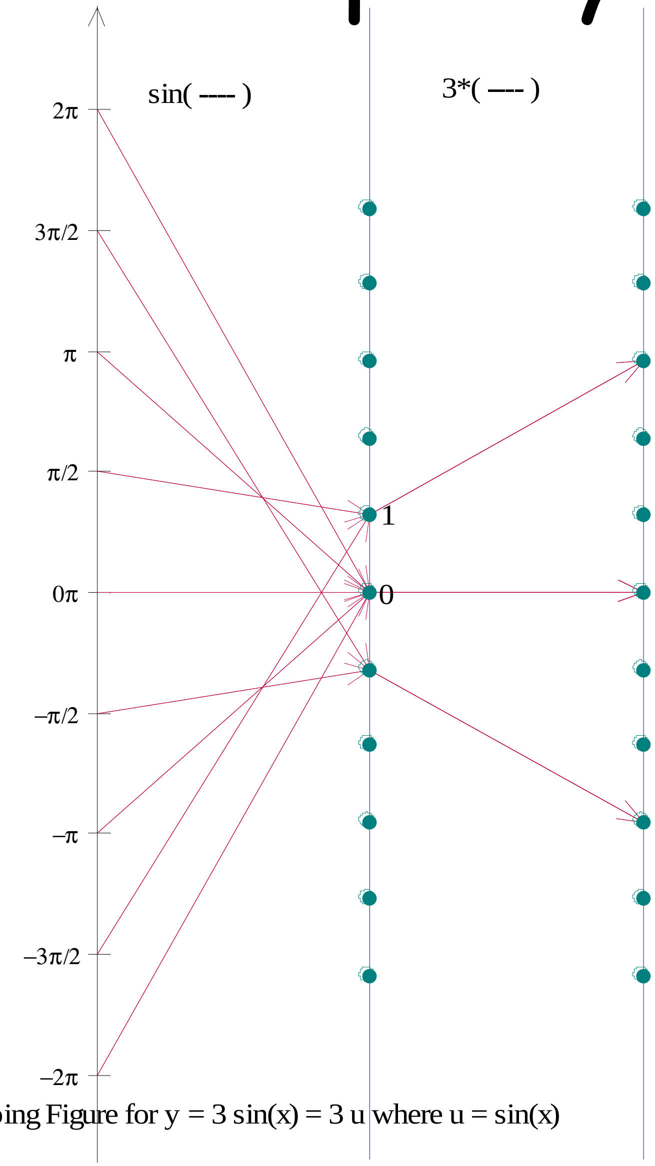
Graph of y = sin(x) + 3 [Winplot]

Amplitude: 1

Period: 2π



Example: $y = f(x) = 3 \sin(x)$



Mapping Figure for y = 3 sin(x) = 3 u where u = sin(x)

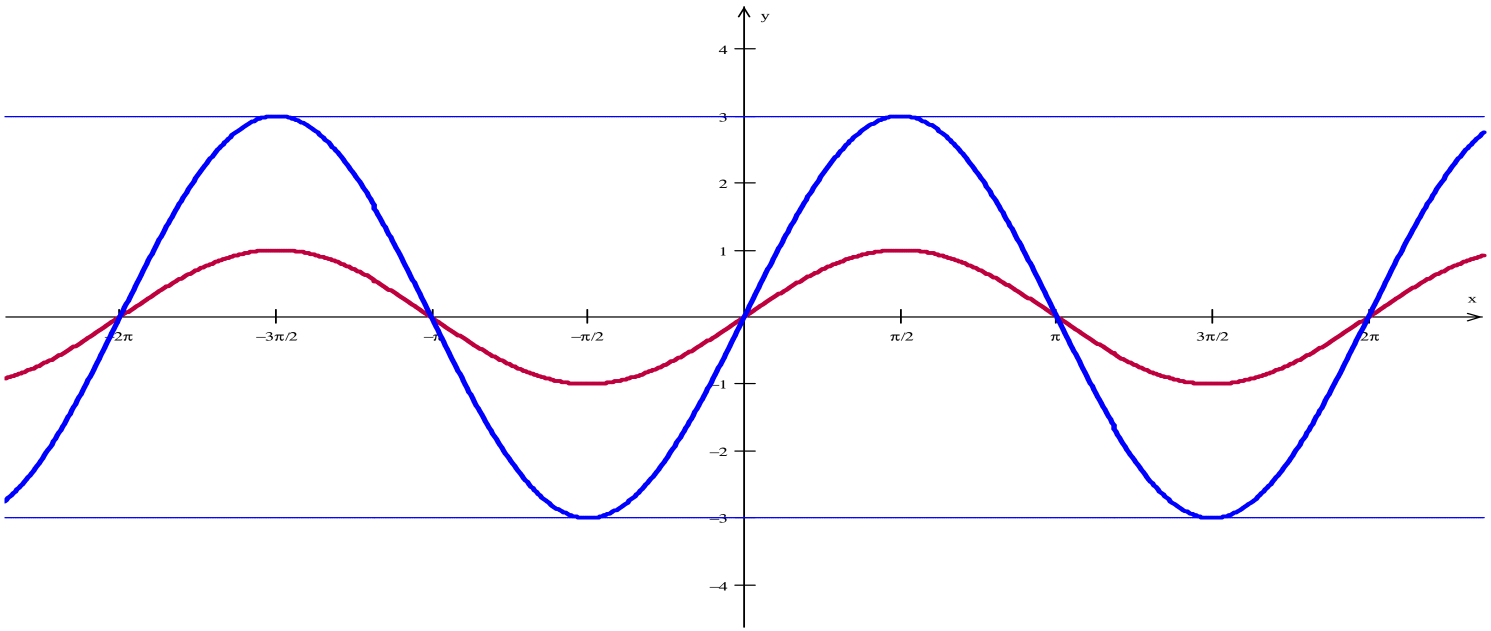
- Mapping diagram for
- $y = f(x) = 3 \sin(x)$ considered as a composition:
- First: $u = \sin(x)$
- Second: $y = 3u$ so the result is
- $y = 3 (\sin(x))$

Example: Graph of y = 3 sin(x)

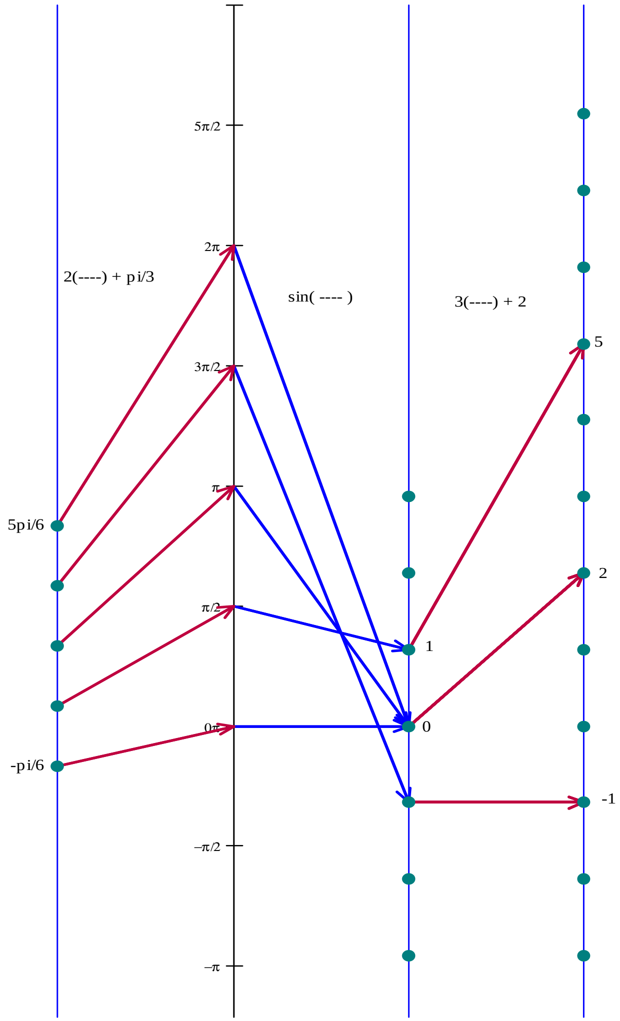
Graph of y = 3 sin(x) [Winplot]

Amplitude: 3

Period: 2π



Mapping diagram



$f(x) = 3 \sin(2x+\pi/3) + 2$

Mapping figure:

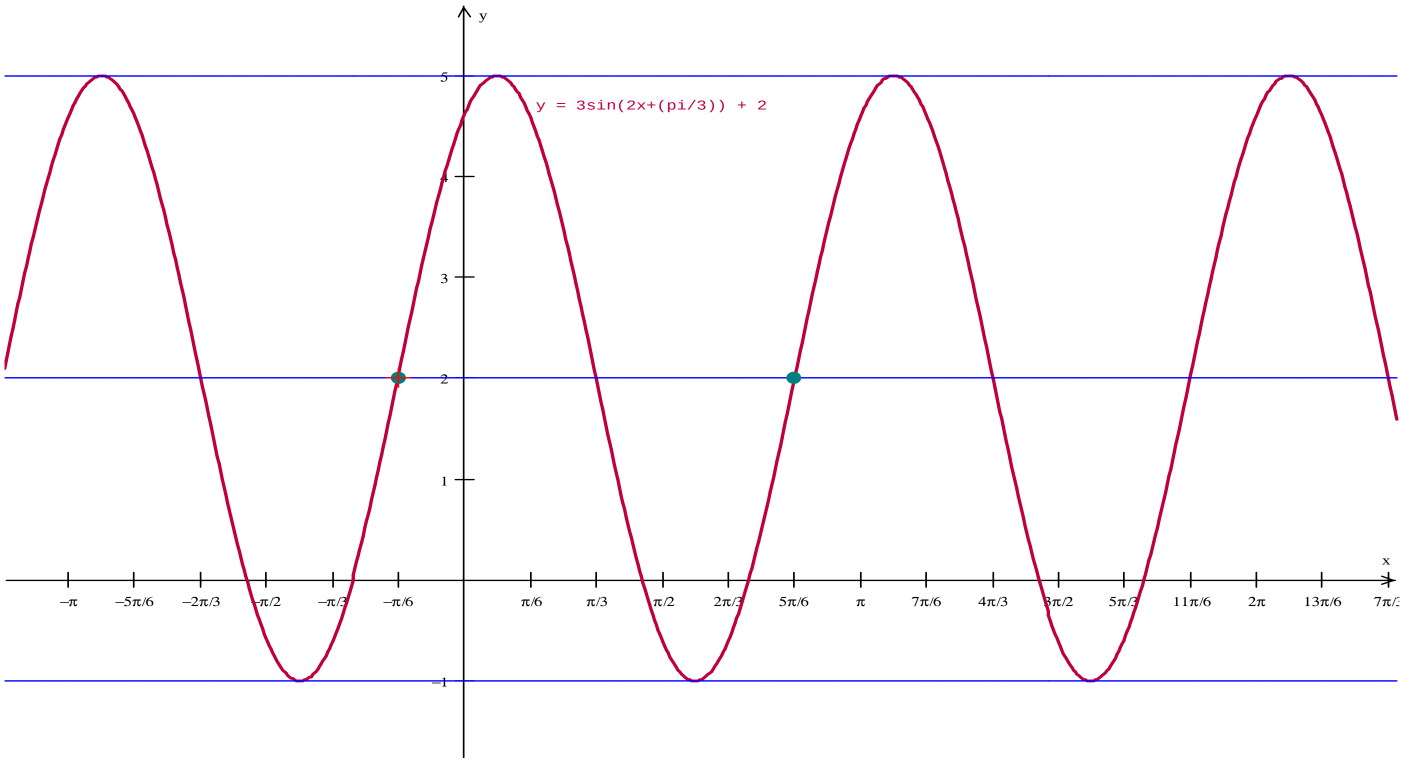
Before $u = 2x+\pi/3$

After $y = 3z + 2$

MIDDLE: $z = \sin(u)$.

Graph

- $f(x) = 3\sin(2x+\pi/3) + 2$



Thanks

The End!

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Questions?

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References

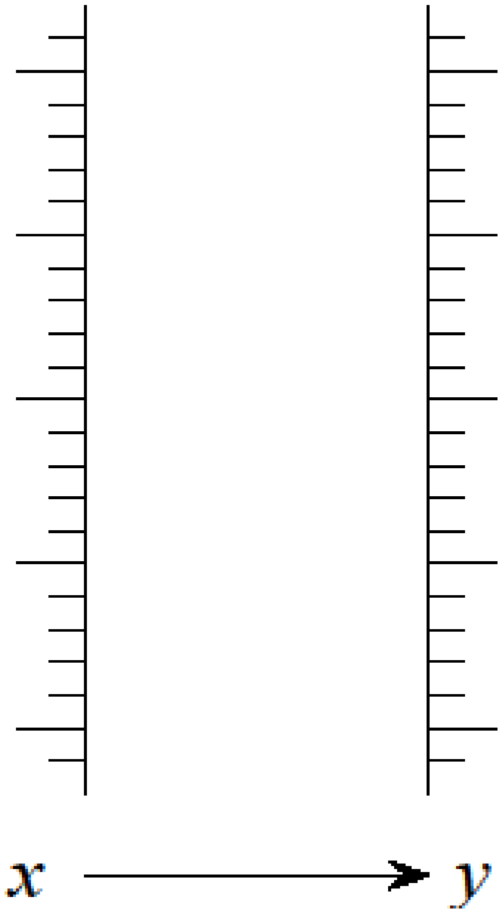
Mapping Diagrams and Functions

- SparkNotes › Math Study Guides › Algebra II: Functions Traditional treatment.
    - http://www.sparknotes.com/math/algebra2/functions/
- **Function Diagrams. by Henri Picciotto Excellent Resources!**
    - **Henri Picciotto's Math Education Page**
    - **Some rights reserved**
- Flashman, Yanosko, Kim https://www.math.duke.edu//education/prep02/teams/prep-12/

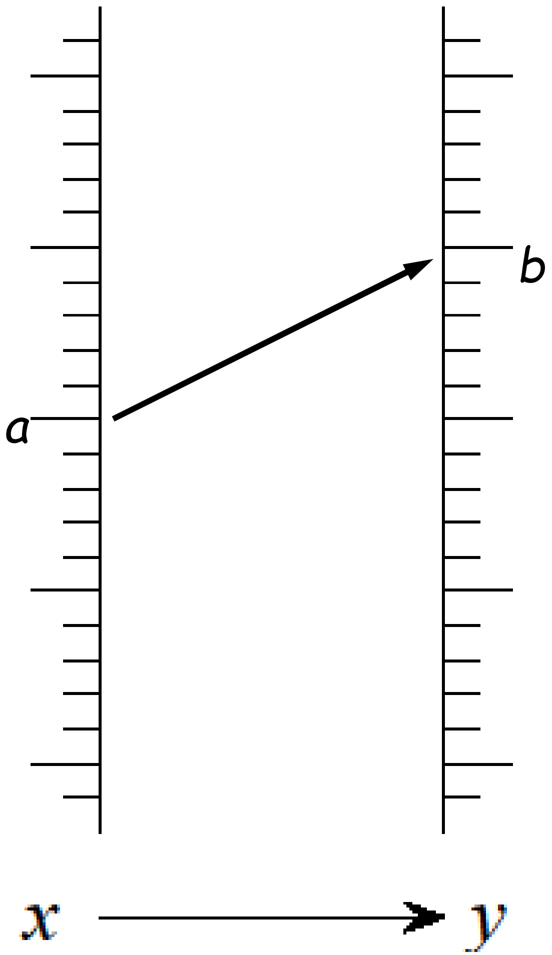
Function Diagrams by Henri Picciotto

Function Diagrams

Henri Picciotto, www.picciotto.org/math-ed



x	y



x	y
a	b

Example: Using Mapping diagrams in “Proof” for Properties of Logs.

$e^{t+x} = e^t e^x = u*y$ where $u = e^t$ and $y = e^x$.

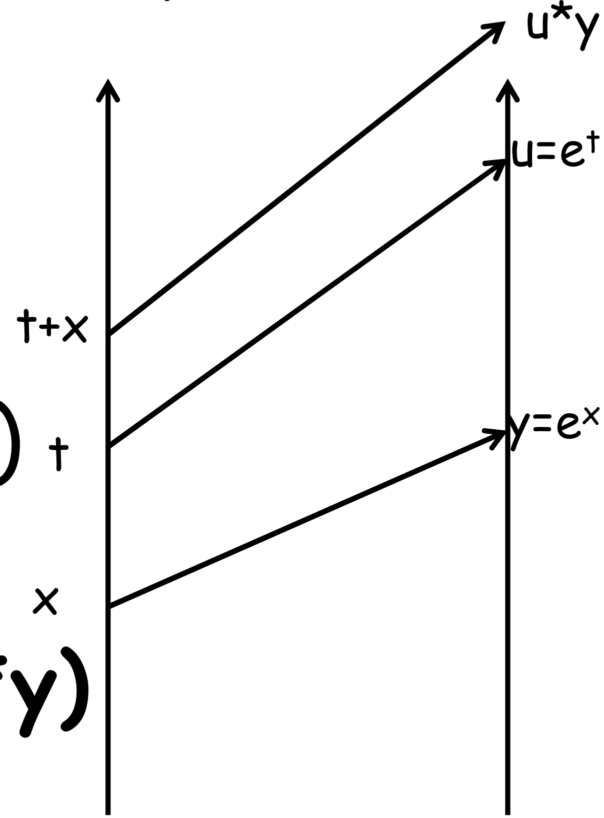
Thus by definition:

$$x = \ln y \; ; \; t = \ln u;$$

And $\quad t+x = \ln(u*y)$

SO

$$\mathbf{\ln u + \ln y = \ln(u*y)}$$



More References

- Goldenberg, Paul, Philip Lewis, and James O'Keefe. "Dynamic Representation and the Development of a Process Understanding of Function." In The Concept of Function: Aspects of Epistemology and Pedagogy, edited by Ed Dubinsky and Guershon Harel, pp. 235–60. MAA Notes no. 25. Washington, D.C.: Mathematical Association of America, 1992.

More References

- http://www.geogebra.org/forum/viewtopic.php?f=2&t=22592&sd=d&start=15

- “Dynagraphs}--helping students visualize function dependency • GeoGebra User Forum

- "degenerated" dynagraph game ("x" and "y" axes are superimposed) in GeoGebra: http://www.uff.br/cdme/c1d/c1d-html/c1d-en.html

Think about These Problems

M.1 How would you use the Linear Focus to **find the mapping diagram for the function inverse for a linear function when m≠0?**

M.2 How does the **choice of axis scales** affect the **position of the linear function focus point** and its use in solving equations?

M.3 Describe the visual features of the mapping diagram for the quadratic function $f(x) = x^2$.
How does this generalize for *even* functions where $f(-x) = f(x)$?

M.4 Describe the visual features of the mapping diagram for the cubic function $f(x) = x^3$.
How does this generalize for *odd* functions where $f(-x) = -f(x)$?

MoreThink about These Problems

L.1 **Describe the visual features of the mapping diagram for the quadratic function** $f(x) = x^2$.
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? “Infinity”?

L.2 Describe the visual features of the mapping diagram for the quadratic function $f(x) = A(x\text{-}h)^2 + k$ using composition with simple linear functions.
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? “Infinity”?

L.3 **Describe the visual features of a mapping diagram for the square root function** $g(x) = \sqrt{x}$ **and relate them to those of the quadratic** $f(x) = x^2$.
Domain? Range? Increasing/Decreasing? Max/Min? Concavity? “Infinity”?

L.4 **Describe the visual features of the mapping diagram for the reciprocal function** $f(x) = 1/x$.
Domain? Range? “Asymptotes” and “infinity”? Function Inverse?

L.5 Describe the visual features of the mapping diagram for the linear fractional function $f(x) = A/(x\text{-}h) + k$ using composition with simple linear functions.
Domain? Range? “Asymptotes” and “infinity”? Function Inverse?

Thanks

The End!   REALLY!

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